The stable classification of 4-manifolds
Krakow
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1 Exercises

1. (a) [1]

   (b) $H_+(\mathbb{Z}) := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

   (c) Special case $M_1 = F_g$ and $M_2 = F_{g'}$: $H_+(\mathbb{Z}) \oplus H_+(\mathbb{Z}^{g+g'})$

2. $\mathrm{sign}(\mathbb{C}P^2) = 1$, all other signatures are 0.

3. (Jan Czajkowski)

   Assume $k \geq 1$.

   First, note that V_k is well-defined in the sense that if $[x_1 : \ldots : x_4] = [y_1 : \ldots : y_4] \in \mathbb{C}P$ then $\sum_i x_i^k = 0 \Leftrightarrow \sum_i y_i^k = 0$.

   V_k is a manifold as a complex submanifold of $\mathbb{C}P$, which I prove below:

   Let $(U_i, \phi_i)_{i=1,\ldots,4}$ (where $\phi_i : U_i \to \mathbb{C}$) be standard atlas for $\mathbb{C}P$:

   $$U_i := \{[x_1 : \ldots : x_4] : x_i \neq 0\},$$

   $$\phi_i([x_1 : \ldots : x_4]) := (\frac{x_1}{x_i}, \ldots, \widehat{\frac{x_i}{x_i}}, \ldots, \frac{x_4}{x_i}).$$

   (Here e. g. $\phi_1([1 : x_2 : x_3 : x_4]) = (x_2, x_3, x_4)$.)

   We will prove that $V_k \cap U_i$ is a submanifold of U_i for $i = 1, \ldots, 4$, which is sufficient for V_k to be a submanifold of $\mathbb{C}P$. Let us focus on the case of $i = 1$ (other cases are proven in the same way).

   Map ϕ_1 is a diffeomorphism between $U_1 \subseteq \mathbb{C}P$ and $\phi_1(U_1) = \mathbb{C}$. Hence we need only to show that $\phi_1(V_k \cap U_1)$ is a submanifold of $\phi_1(U_1) = \mathbb{C}$. But

   $$\begin{aligned} \phi_1(V_k \cap U_1) &= \{(x_2, x_3, x_4) \in \mathbb{C} : \phi_1^{-1}(x_2, x_3, x_4) = [1 : x_2 : x_3 : x_4] \in V_k \cap U_1\} \\ &= \{(x_2, x_3, x_4) \in \mathbb{C} : 1 + x_2^k + x_3^k + x_4^k = 0\}. \end{aligned}$$

Let define $f : \mathbb{C} \to \mathbb{C}$: $f(x_2, x_3, x_4) := 1 + x_2^k + x_3^k + x_4^k$ (it is a holomorphic function). We show that 0 is a regular value of f: $\mathrm{D}f(x_2, x_3, x_4) = (kx_2^{k-1}, kx_3^{k-1}, kx_4^{k-1})$ is not of maximal order (which means equal to 0) only if $(x_2, x_3, x_4) = 0$, but then $f(x_2, x_3, x_4) = 1 \neq 0$. So if $f(x_2, x_3, x_4) = 0$, then $\mathrm{D}f(x_2, x_3, x_4)$ is of maximal order, so 0 is a regular value of f.

Hence $\phi_1(V_k \cap U_1) = f^{-1}(0)$ is a submanifold of $\mathbb{C}$; that finishes the proof.

4. ...

5. * ...

6. (Michal Marcinkowski) Since b is non-singular (unimodular) we have the isomorphism $b^* : A \to A^* = Hom(A, Z)$ st. $b^*(\alpha) = \alpha^*$ where $\alpha^*(\beta) = b(\alpha, \beta)$.

   Choose a primitive element $\alpha_1 \in C$. Since α^* is primitive in A^*, it is ephimorphism. Thus we can find β_1 st. $b(\alpha_1, \beta_1) = 1$ (automatically β_1 is primitive and we can think of α_1 and β_1 as of a base elements). Define $V = \{x \in A : b(\alpha_1, x) = 0 = b(\beta_1, x)\}$. By straightforward computations we have that $A = V \oplus Z[\alpha_1, \beta_1]$ (so $rank(V) = rank(A) - 2$) and that $rank(C \cap V) = rank(C) - 1$ (Choose the basis $\{\alpha_i\}$, $i = 1, \ldots, n$ of C, then one can find the numbers $\{k_j\}$, $j = 2, \ldots, n$ such that $\{\alpha_j - k_j\alpha_1\}$, $j = 2, \ldots, n$ is the basis of $C \cap V$, which is still a summand).

   Notice now that $b|_V$ is non-singular (by def. of V) and $C \cap V$ is a Lagrangian of $b|_V$. Proceeding by induction we obtain the basis $\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots$ st. the matrix of b in this basis has the form:

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & & & & \\ 1 & m_1 & 0 & \cdots & & & & \\ 0 & 0 & 0 & 1 & 0 & \cdots & & \\ \vdots & \vdots & 1 & m_2 & 0 & \cdots & & \\ & & & & \ddots & & & \\ & & & & & & 0 & 1 \\ & & & & & & 1 & m_n \end{pmatrix}$$

   where $m_j = b(\beta_j, \beta_j)$. Now it is easy to see that since the form $\left(\begin{smallmatrix} 0 & 1 \\ 1 & m \end{smallmatrix}\right)$ has a signature zero, then b has a signature zero.

7. Consider the maps $\partial W \xrightarrow{i} W \xrightarrow{q} (W, \partial W)$ and (co-)homology with $\mathbb{Q}-$coefficients. Claim: $i^* H^{2k}(W) \subset H^{2k}(\partial W)$ is the Lagrangian we are looking for.

   Proof. There are two things we need to show:

- the cup product pairing vanishes on $i^*H^{2k}(W)$
- $\mathrm{dim} i^*H^{2k}(W) = 1/2 \mathrm{dim} H^{2k}(\partial W)$.

To see the first thing consider two elements $i^*\alpha, i^*\beta$ and consider the cup product pairing:

$$\langle i^*\alpha \cup i^*\beta, [\partial W]\rangle = \langle i^*(\alpha \cup \beta), [\partial W]\rangle = \langle \alpha \cup \beta, i_*[\partial W]\rangle.$$

This vanishes because $i_*[\partial W] = 0$.

Now it remains to show that $i^*H^{2k}(W)$ is a half dimensional subspace. To see that consider the following commutative diagram, in which PD respectively LD denote Poincaré and Lefschetz duality.

$$\begin{array}{ccccc} H^{2k}(W) & \xrightarrow{i^*} & H^{2k}(\partial W) & \xrightarrow{\delta} & H^{2k+1}(W, \partial W) \\ \mathrm{LD}\downarrow\cong & & \mathrm{PD}\downarrow\cong & & \mathrm{LD}\downarrow\cong \\ H_{2k+1}(W, \partial W) & \xrightarrow{\partial} & H_{2k}(\partial W) & \xrightarrow{i_*} & H_{2k}(W) \end{array}$$

Note that the universal coefficient theorem for fields gives an isomorphism between $H_{2k,2k+1}(W, \partial W)$ and $H^{2k,2k+1}(W, \partial W)$, respectively.
Now we will use the fact that $\mathrm{coker} f \cong \ker f^*$ for some map $f : V \to W$ between finite vector spaces and f^* the induced map on the dual spaces.
We get:

$$\begin{aligned} dim H_{2k}(\partial W) &= dim \ker i_* + dim \mathrm{im} i_* = dim \mathrm{coker} i^* + dim \mathrm{im} i_* \\ &= dim H^{2k}(\partial W) - dim \mathrm{im} i^* + dim \mathrm{im} i_* \\ &\Rightarrow dim \mathrm{im} i_* = dim \mathrm{im} i^*. \end{aligned}$$

One part of the proof of Lefschetz duality is that the squares in the above diagram commute up to sign. One can see that by considering simplicial homology, a triangulation of W and checking that the maps agree on chain level.
From the commutativity we know that $dim \ker i_* = dim \ker \delta = dim \mathrm{im} i^*$. Combined we get

$$\begin{aligned} \dim H_{2k}(\partial W) &= \dim \ker i_* + \dim \mathrm{im} i_* = \dim \ker \delta + \dim \mathrm{im} i_* \\ &= \dim \mathrm{im} i^* + \dim \mathrm{im} i^* \end{aligned}$$

as we wanted to show. □

2 Tuesday

1. (Anna Abczynski)

   Before we start to consider the actual exercise we want to understand why the following formula for smooth $2m-$, respectively $2n-$dimensional manifolds M^{2m} and N^{2n} holds:

$$sign(M \times N) = sign(M) sign(N).$$

Through the proof we assume $n \leq m$ and all cohomology is understood as cohomology with rational coefficients. The crucial ingredient for the proof is the Künneth theorem stating

Theorem 3. *The cross product*

$$\begin{aligned} H^*(X_1;R) \otimes H^*(X_2;R) &\overset{\times}{\to} H^*(X_1 \times X_2;R) \\ a \otimes b &\mapsto p_1^*(a) \cup p_2^*(b) \end{aligned}$$

is an isomorphism of rings if X_1 and X_2 are CW complexes, $p_i : X_1 \times X_2 \to X_i$ are the projections upon the first, respectively second factor and $H^k(X_2;R)$ is a finitely generated free R module for all k.

Assuming our cohomology ring is finitely generated we can decompose the cohomology ring of $M \times N$ i.p. in the middle dimension $m+n$ where

$$H^{m+n}(X \times Y) \cong H^n(N) \otimes H^m(M) \oplus \underbrace{\bigoplus_{i=-n, i\neq 0}^{n} H^{n-i}(N) \otimes H^{m+i}(M)}_{=:V}.$$

Start with considering the cup product pairing

$$H^{n-i}(N) \otimes H^{m+i}(M) \times H^{n-j}(N) \otimes H^{m+j}(M) \to \mathbb{Q},$$

where $i, j \in \mathbb{Z}, i \neq -j$ and one of them is non vanishing. We get

$$a \otimes b \cup a' \otimes b' \overset{\times}{\mapsto} p_1^*(a \cup a') \cup p_2^*(b \cup b').$$

Then $a \cup a'$ is a $(2n-i-j)-$form and $b \cup b'$ is a $2m+i+j$-form and therefore one of them vanishes meaning that the cup product pairing vanishes on all these factors leaving only factors of the form

$$H^{n-i}(N) \otimes H^{m+i}(M) \times H^{n+i}(N) \otimes H^{m-i}(M), i = 0, ..., n$$

to consider.
After choosing a basis for $H^{n-i}(N) \otimes H^{m+i}(M)$ and $H^{n+i}(N) \otimes H^{m-i}(M)$ respectively we can represent the cup product pairing of the both by some quadratic matrix U_i. The matrix is quadratic because $H^{m-i}(M) \cong H_{m+i}(M) \cong H^{m+i}(M)$ by Poincaré duality and the universal coefficient theorem. The same holds for N.
Combining these statements the intersection form $\lambda_{N\times M}$ of $M \times N$, where λ_M and λ_N are the intersection forms of M and N, can be represented by the following matrix:

$$\begin{pmatrix} \lambda_M \otimes \lambda_N & 0 & ... & 0 & 0 & ... & 0 \\ 0 & 0 & ... & 0 & U_1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 & \ddots & 0 \\ 0 & 0 & ... & 0 & 0 & 0 & U_{2n} \\ 0 & U_1^t & 0 & 0 & 0 & ... & 0 \\ \vdots & 0 & \ddots & 0 & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & U_{2n}^t & 0 & ... & 0 \end{pmatrix}.$$

In this form we can directly see that there is a half rank subspace L of V with vanishing intersection form $\lambda_{N\times M}|_L \equiv 0$. Hence $sign\lambda_{N\times M}|_V = 0$.
Since $\lambda_{M\times N} = \lambda_M \otimes \lambda_N \perp \lambda_{N\times M}|_L$ and owing to the fact that the signature of the tensor product of two symmetric bilinear forms over $\mathbb{Q}$ is the product of their signatures we showed that $sign(M\times N) = sign(M)sign(N)$.

The exercise now follows as an easy example. We know that

$$sign(\mathbb{C}P^m) = \begin{cases} 1 & m \text{ even} \\ 0 & m \text{ odd.} \end{cases}$$

Therefore

$$sign(\mathbb{C}P^m \times \mathbb{C}P^n) = \begin{cases} 1 & n \text{ and } m \text{ even} \\ 0 & \text{else.} \end{cases}$$

2. (Marek Kaluba)

   Show that the stable complex tangent bundle of $\mathbb{C}P^n$ is $[(n+1)\bar{L}]$, where $\bar{L}$ is the conjugate of tautological bundle over $\mathbb{C}P^n$ and $(n+1)\bar{L}$ denotes $(n+1)$-fold Whitney sum of $\bar{L}$ with itself.

Fact: The tangent bundle to $\mathbb{C}P^n$ is isomorphic to $\hom(L, L^\perp)$.

Let A be a complex line in $\mathbb{C}^{n+1}$, intersecting S^{2n+1} along the circle S, and $A^\perp$ be the orthogonal subspace to A in $\mathbb{C}^{n+1}$. Let $f\colon S^{2n+1} \to \mathbb{C}P^n$ be the canonical projection, $f(z) = \{\lambda z\colon \lambda \in S^1\}$. Note that the all pairs of form $(\lambda z, \lambda v)$ in TS^{2n+1}, the tangent bundle to the sphere, have the same image under the map

$$Df\colon TS^{2n+1} \to T\mathbb{C}P^n$$

induced by f. Thus tangent bundle $T\mathbb{C}P^n$ can be identified with the set of sets $\{(\lambda z, \lambda v)\colon \lambda \in S^1\}$ (which are equivalence classes of points in $\mathbb{C}P^n$ and vectors tangent to them) satisfying

$$z\cdot\bar{z} = 1, \qquad z\cdot v = 0.$$

Each such class determines, and is determined by, a linear mapping $\ell\colon A \to A^\perp$ given by

$$\ell(z) = v.$$

Thus tangent space of $\mathbb{C}P^n$ at $\{\lambda z\colon \lambda \in S^1\}$ is isomorphic to the vector space $\hom(A, A^\perp)$. Because the isomorphism is canonical, it follows, that vector bundle $T\mathbb{C}P^n$ is isomorphic to the bundle $\hom(L, L^\perp)$.

Now we can use this fact to compute stable tangent bundle of $\mathbb{C}P^n$. In the rest of the proof, ε^2 denotes 1-dimensional, trivial complex vector bundle. The bundle $\hom(L, L)$ is trivial, since it has nowhere zero-section (which has also non-zero real and imaginary parts). Therefore

$$T\mathbb{C}P^n \oplus \varepsilon^2 \cong \hom(L, L^\perp) \oplus \hom(L, L) \cong \hom(L, \varepsilon^{2n+2}.$$

This is clearly isomorphic to

$$\hom(L, \varepsilon^2 \oplus \cdots \oplus \varepsilon^2) \cong \underbrace{\hom(L, \varepsilon^2) \oplus \cdots \oplus \hom(L, \varepsilon^2)}_{n+1 \text{ times}}.$$

But bundle $\hom(L, \varepsilon^2)$ is isomorphic to $\bar{L}$. This proves that

$$T\mathbb{C}P^n \oplus \varepsilon^2 \cong \underbrace{\bar{L} \oplus \cdots \bar{L}}_{n+1 \text{ times}}.$$

3. (Philipp Kühl)

From exercise 2 we know:

$$c(\mathbb{C}P^n) = c((n+1)\bar{L}) = c(\bar{L})^{n+1} = (1-x)^{n+1}$$

where $x \in H^2(\mathbb{C}P^n)$ is a generator. So

$$c(\mathbb{C}P^n \otimes \mathbb{C}) = c(\mathbb{C}P^n \oplus \overline{\mathbb{C}P^n}) = (1-x)^{n+1}(1+x)^{n+1} = (1-x^2)^{n+1}$$

Since $p_i(\mathbb{C}P^n) = (-1)^i c_i(\mathbb{C}P^n \times \mathbb{C})$ it follows

$$p(\mathbb{C}P^n) = (1+x^2)^{n+1}$$

4. We check the Pontryagin number p_{m+n}. Using Tuesday problem 3, the Cartan product formula for the rational Pontryagin classes and the fact that $p_r \in H^{4r}$ must evaluate to 0 if $4r$ is greater than the dimension of the underlying manifold, we get

$$\langle p_{m+n}(\mathbb{C}P^{2m} \times \mathbb{C}P^{2n}), [\mathbb{C}P^{2m} \times \mathbb{C}P^{2n}] \rangle = \langle p_m(\mathbb{C}P^{2m}), [\mathbb{C}P^{2m}] \rangle \langle p_n(\mathbb{C}P^{2n}), [\mathbb{C}P^{2n}] \rangle$$

$$\langle p_m(\mathbb{C}P^{2m}), [\mathbb{C}P^{2m}] \rangle \langle p_n(\mathbb{C}P^{2n}), [\mathbb{C}P^{2n}] \rangle = \binom{2m+1}{m}\binom{2n+1}{n}$$

and

$$\langle p_{m+n}(\mathbb{C}P^{2m+2n}), [\mathbb{C}P^{2m+2n}] \rangle = \binom{2m+2n+1}{m+n}.$$

Assume now that $n \geq m$. Then increasing n by 1 and decreasing m by 1 in the formulae above multiplies the result by

$$\frac{\binom{2m-1}{m-1}\binom{2n+3}{n+1}}{\binom{2m+1}{m}\binom{2n+1}{n}} = \frac{(2n+3)(2n+2)m(m+1)}{(n+1)(n+2)(2m+1)2m} = \frac{2 - \frac{1}{n+2}}{2 - \frac{1}{m+1}} > 1.$$

Repeat this until reaching $m+n$ and 0, and we get that

$$\binom{2n+1}{n}\binom{2m+1}{m} < \binom{2n+3}{n+1}\binom{2m-1}{m-1} < \cdots < \binom{2n+2m-1}{m+n},$$

hence the characteristic numbers p_{m+n} are indeed different for our two manifolds.

Less sensible version: Same strategy, the characteristic number is p_1^{m+n}. We have

$$\begin{aligned}&\langle p_1^{m+n}(\mathbb{C}P^{2m}\times\mathbb{C}P^{2n}),[\mathbb{C}P^{2m}\times\mathbb{C}P^{2n}]\rangle=\\&\langle((2n+1)x\otimes 1+(2m+1)1\otimes x)^{m+n},[\mathbb{C}P^{2m}\times\mathbb{C}P^{2n}]\rangle=\\&\tbinom{m+n}{n}(2n+1)^n\tbinom{m+n}{m}(2m+1)^m\end{aligned}$$

and

$$\langle p_1^{m+n}(\mathbb{C}P^{2m+2n}),[\mathbb{C}P^{2m+2n}]\rangle=(2m+2n+1)^{m+n}.$$

We apply $\binom{m+n}{n}>(1+\frac{m}{n})^n$ (all multiplied fractions are at least $\frac{m+n}{n}$, and only the first is exactly equal) to get the estimates

$$\binom{m+n}{n}(2n+1)^n>\left(2m+2n+\frac{m+n}{n}\right)^n>(2m+2n+1)^n$$

and

$$\binom{m+n}{m}(2m+1)^m>\left(2m+2n+\frac{m+n}{m}\right)>(2m+2n+1)^m.$$

The product of these shows that p_1^{m+n} differs on our two manifolds.

Lumberjack version: Forget signature, compute p_1^{m+n} and $p_2p_1^{m+n-2}$. When the red fog clears, one sees that the two numbers are not the same.

5. (Lukasz Bak)

Clearly, the polynomial L must be of a form

$$L(p_1,p_2)=ap_1^2+bp_2, \tag{1}$$

ie. the signature of a manifold is a linear combination of its Pontrjagin numbers. Since Pontrjagin numbers are bordism invariants, to compute a and b we have to evaluate this polynomial on two manifolds linearly independent in Ω_8^{SO}, a ring of oriented bordism classes of closed oriented smooth 8-dimensional manifolds. Thanks to Thom we know that

$$\Omega_*\otimes\mathbb{Q}=\mathbb{Q}[\mathbb{C}P^2,\mathbb{C}P^4,\ldots],$$

so in particular $\mathbb{C}P^2\times\mathbb{C}P^2$ and $\mathbb{C}P^4$ are linearly independent.

Now

$$\begin{aligned}c(\mathbb{C}P^2\times\mathbb{C}P^2)&=(1-x_1)^3(1-x_2)^3,\\c(\mathbb{C}P^4)&=(1-x)^5,\end{aligned}$$

where x, x_1 and x_2 are first Chern classes of canonical bundles over $\mathbb{C}P^4$ and respective copies of $\mathbb{C}P^2$. Thus

$$\begin{aligned}p(\mathbb{C}P^2\times\mathbb{C}P^2)&=(1+x_1^2)^3(1+x_2^2)^3=1+3x_1^2+3x_2^2+9x_1^2x_2^2,\\p(\mathbb{C}P^4)&=(1+x^2)^5=1+5x^2+10x^4,\end{aligned}$$

and finally

$$\begin{aligned} p_1^2(\mathbb{C}P^2 \times \mathbb{C}P^2) &= 18x_1^2x_2^2, \\ p_2(\mathbb{C}P2 \times \mathbb{C}P^2) &= 9x_1^2x_2^2, \\ p_1^2(\mathbb{C}P^4) &= 25x^4, \\ p_2(\mathbb{C}P^4) &= 10x^4. \end{aligned}$$

Using this, the equality (1) and $\text{sign}(\mathbb{C}P^2 \times \mathbb{C}P^2) = \text{sign}(\mathbb{C}P^4) = 1$ we obtain

$$\begin{cases} 18a + 9b &= 1 \\ 25a + 10b &= 1 \end{cases}$$

with the unique solution $a = -\frac{1}{45}$, $b = \frac{7}{45}$.

6. (Semra Pamuk)

   Assume $n \geq 3$.

   Let A be a collar neighbourhood of one of the boundary components and $B := W - A$. By Seifert- Van Kampen theorem,

$$\pi_1(W) \cong (\pi_1(A) * \pi_1(B))/\sim$$

   since both A and B are 1-connected, $\pi_1(W) \cong 0$. Now from Mayer-Vietoris sequence

$$\cdots \longrightarrow H_i(A \cap B) \longrightarrow H_i(A) \oplus H_i(B) \longrightarrow H_i(W) \longrightarrow H_{i-1}(A \cap B) \longrightarrow \cdots$$

   for $i \neq n-1$; $H_i(W) = 0$. For $i = n-1$; we have

$$\cdots \longrightarrow H_{n-1}(A \cap B) \longrightarrow H_{n-1}(A) \oplus H_{n-1}(B) \longrightarrow H_{n-1}(W) \longrightarrow 0$$

   and $H_{n-1}(A \cap B) \cong H_{n-1}(A) \cong \mathbb{Z}$ also $H_{n-1}(B) \cong H_{n-1}(S^{n-1}) \cong \mathbb{Z}$. Now by Whitehead theorem, $W \simeq S^{n-1}$ and hence it is an h-cobordism.

7. ...

8. (Wojtek Politarczyk)
   Consider $X^4 := \mathbb{C}P^2 - (intD_1^4 \cup intD_2^4)$, where D_1^4 and D_2^4 are disjoint embedded disks in $\mathbb{C}P^2$. Suppose X is bordant rel. boundary to an h-cobordism. Thus there exists 5-manifold W^5 whose boundary is

$$\partial W^5 = X^4 \cup S^3 \times [0,1] \cup S^3 \times [0,1] \cup Y^4,$$

   where Y^4 is an h-cobordism between S^3 and S^3. $S^3 \times [0,1] \cup S^3 \times [0,1] \cup Y^4$ has the same cohomology as $S^3 \times [0,1]$ thus ∂W^5 has the same cohomology as $\mathbb{C}P^2 \# S^1 \times S^3$. We can compute signature of ∂W^5

$$\sigma(\partial W^5) = \sigma(\mathbb{C}P^2) + \sigma(S^1 \times S^3).$$

   Exercise 1 from Monday tells us that $\sigma(S^1 \times S^3) = 0$ thus $\sigma(\partial W^5) = 1$ which is a contradiction. It follows that X^4 can't be bordant rel. boundary to an h-cobordism.

4 Wednesday

1. (Philipp Kühl)

   The long exact homotopy sequence

   $$\ldots \to \pi_{i+1}(S^n) \to \pi_i(\mathrm{SO}_n) \to \pi_i(\mathrm{SO}_{n+1}) \to \pi_i(S^n) \to \pi_{n-1}(\mathrm{SO}_n) \to \ldots$$

   gives us

   $$0 \to \pi_i(\mathrm{SO}_n) \xrightarrow{\cong} \pi_i(\mathrm{SO}_n + 1) \to 0 \quad \text{for } i < n-1$$

   since $\pi_i(S^n) = 0$ for $i < n$.

2. (Mehmetcik Pamuk)

   We have

   $$\det\colon \pi_0(O_n) \xrightarrow{\cong} \mathbb{Z}/2 \ ,$$

   hence $\pi_0(O) \cong \mathbb{Z}/2$.

   For $n \geq 3$; by question 1, $\pi_1(SO_n) \cong \pi_1(SO_3) \cong \mathbb{Z}/2$, where $SO_3 \cong \mathbb{R}P^3$. Hence we have $\pi_1(SO) \cong \mathbb{Z}/2$.

   For $\pi_2(SO)$, consider

   $$\pi_2(SO_3) \longrightarrow \pi_2(SO_4) \longrightarrow \pi_2(S^3) \ .$$

   Since $\pi_2(SO_3) \cong \pi_2(S^3) \cong 0$, we have $\pi_2(SO_4) \cong 0$. Again by question 1, $\pi_2(SO) \cong \pi_2(SO_4) \cong 0$.

   Actually by Bott Periodicity, we know that the homotopy groups of SO repeat periodically with a period of 8 and

   $$\pi_i(SO) \cong \mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z} \quad \text{for} \quad 0 \leq i \leq 7 \ .$$

3. (Michal Marcinko)

   Assume $l < m-1$. Considering the injection $SO(m) \to SO(m+1)$ we see that the isomorphism $i_m : \pi_l(SO(m)) \to \pi_l(SO(m+1))$ is given by $i_m(E) = E \oplus R$ (see Ex. 1). Notice that $Vect_{R^k}(S^r) = \pi_{r-1}(SO(k)) = \pi_{r-1}(SO)$.

   We see from the above that the isomorphism $j : \pi_{r-1}(SO(k)) \to \pi_{r-1}(SO(k+n))$ (j is just a composition of i's) maps E to $E \oplus R^n$ which is the trivial bundle, so E is also trivial. We can choose a trivialisation and write simply $E = R^n$

   Any (orthonormal) trivialisation of the bundle $E \oplus R^n = R^k \oplus R^n$ is a map $S^r \to SO(k+n)$, so any homotopy class of such a trivialisation is an element of $\pi_r(SO(k+n)$. If we assume that $r < k-1$ we see that $\pi_r(SO(k+n)) \simeq \pi_r(SO(k))$. It follows that any homotopy class of a trivialisation of $E \oplus R^n$ gives us a homotopy class of a trivialisation of E.

4. *Solution (Lukasz Bak).* The nontrivial implication is from right to left. Assume that $E \oplus \underline{\mathbb{R}}$ is trivial and $e(E) = 0$. The classifying map $f' : S^{2k-1} \to SO_{2k+1}$ of $E \oplus \underline{\mathbb{R}}$ factors through the classifying map $f : S^{2k-1} \to SO_{2k}$ of E,

$$\begin{array}{ccc} SO_{2k} & \xrightarrow{i} & SO_{2k+1} \\ {\scriptstyle f}\uparrow & \nearrow{\scriptstyle f'} & \\ S^{2k-1} & & \end{array} .$$

Let $x \in \pi_{2k-1}(SO_{2k})$ be the class represented by f. Clearly $i_* x = 0$. We need to show that $x = 0$. From the homotopy long exact sequence of the fibration $SO_{2k} \to SO_{2k+1} \to S^{2k}$,

$$\ldots \to \pi_{2k}(S^{2k}) \xrightarrow{\partial} \pi_{2k-1}(SO_{2k}) \xrightarrow{i_*} \pi_{2k-1}(SO_{2k+1}) \to \pi_{2k-1}(S^{2k}) \to \ldots,$$

and the hint above, we see that $x = a[TS^{2k}]$, where $[TS^{2k}]$ is a homotopy class represented by a clutching function of the tangent bundle. But then

$$0 = e(E) = ae(TS^{2k}) = 2a \in H^{2k}(S^{2k}; \mathbb{Z}) \cong \mathbb{Z}$$

and hence $a = 0$. □

5 Thursday

1. (Marek Kaluba)

   Let (X, E) be a vector bundle, and let $h \colon S^{i+1} \to X$ be a map. Let $F \to S^{i+1}$ be a vector bundle such that F and $-h^*E$ are stably isomorphic. Show that $S(F)$, the sphere bundle of F admits a normal structure in (X, E).

   To show existence of a normal structure we need to specify a continues map

$$f \colon S(F) \to X,$$

   and show that $\nu(S(F)) \cong f^*E$ are isomorphic as a vector bundles.

   Since we don't have much choice, let's define $f = \pi_F \circ h$. The pullback bundle via f can be described as follows.

$$f^*E = \pi_F^*(h^*E) = -\pi_F^* F,$$

   where the equalities are meant to be stable isomorphisms.

   Choose an embedding of the manifold $S(F)$ into some $\mathbb{R}^n$. Since $T(S(F)) \oplus \nu(S(F) \hookrightarrow \mathbb{R}^n) \cong \varepsilon^n_{\mathbb{R}^n}$, what we really need to show is an isomorphism

$$f^*(E) \oplus T(S(F)) \cong \varepsilon^k_{\mathbb{R}^k},$$

or differently speaking, that stable class $[f^*E]$ is an inverse of $[T(S(F))]$ in stable K-theory of S^{i+1}.

Observe that π_F is a restriction of projection $\pi\colon TF \to S^{i+1}$ to the zero section. Hence $TF|_{S^{i+1}} \cong TS^{i+1} \oplus \nu(S^{i+1} \hookrightarrow F)$, and since the last summand is isomorphic to F we have

$$TF|_{S^{i+1}} \cong TS^{i+1} \oplus F.$$

After pulling back via π we obtain

$$T(S(F))|_{S^{i+1}} = \pi^* TS^{i+1} \oplus \pi^*(F).$$

Finally we have following isomorphisms up to addition of a trivial summand:

$$TS(F)|_{S^{i+1}} \oplus f^*E \oplus \varepsilon_{S^{i+1}} \cong \pi^* TS^{i+1} \oplus \varepsilon_{S^{i+1}} \oplus \pi^* F|_{S^{i+1}} \oplus -\pi_F^* F \cong$$
$$\pi^*(TS^{i+1} \oplus \varepsilon_{S^{i+1}}) \oplus \pi_F^* F \oplus -\pi_F^* F \cong \varepsilon_{S^{i+1}}.$$

2. Denote by r the greatest integer for which $\pi_r(F) : \pi_r(W) \to \pi_r(X)$ is not an isomorphism; if such an integer does not exist or $r \geq [n/2]$, we are done. We perform surgery on (W, F, β) to incrementally improve r. First assume that $r \geq 1$. Then $\pi_j(X, W) = 0$ for $j \leq r$, therefore by (relative) Hurewicz theorem $H_j(X, W) = 0$ for $j \leq r$ and $H_{r+1}(X, W) \cong \pi_{r+1}(X, W)$. Consider the following fragment of the homology long exact sequence of the pair (X, W):

$$H_{r+1}(W) \to H_{r+1}(X) \to H_{r+1}(X, W) \to H_r(W) \to H_r(X) \to H_r(X, W) = 0.$$

We first kill the kernel $K := \ker H_r(F) : H_r(W) \to H_r(X)$. Given an element $\alpha \in K$, it is mapped to 0 in the exact sequence above, so it is the image under the boundary map of some $\hat{\alpha} \in H_{r+1}(X, W) \cong \pi_{r+1}(X, W)$. Hence a spherical representative of $\hat{\alpha}$ can be chosen:

$$\hat{\alpha} = [((D^{r+1}, S^r), g)]$$

and

$$\alpha = [(S^r, g|_{S^r})].$$

Without loss of generality we may assume that $\partial g := g|_{S^r} : S^r \to W$ is an embedding if $2r < \dim W = n + 1$. The map $\partial g := g|_{S^r} : S^r \to X$ is null-homotopic, therefore the stable normal bundle of W restricted to the range of S^r is trivial as the pullback of E by ∂g. The normal bundle of ∂g is the difference of this stably trivial bundle and the stably trivial bundle TS^r, therefore it is stably trivial. Additionally its rank is $n+1-r > r$, so this normal bundle is trivial. Performing a surgery on the embedding ∂g kills α.

After making K trivial, we make $H_{r+1}(F) : H_{r+1}(W) \to H_{r+1}(X)$ surjective, proceeding one generator α of $H_{r+1}(X, W) \cong \pi_{r+1}(X, W)$ at a time. Note that the map $\pi_r(F) : \pi_r(W) \to \pi_r(X)$ is an isomorphism as the maps $H_j(F)$ are isomorphisms for $j \leq r$; consequently the boundary map is identically 0 on $\pi_{r+1}(X, W)$, all classes here lift to $\pi_{r+1}(X)$. We may hence assume that α is represented by a map $h : S^{r+1} \to X$.

Consider now Thursday exercise 1 in the setting $F = -h^*(E)$ with rank $n-r+1 > r+1$; the conclusion of the exercise is that $S(F)$ admits a normal (X, E)-structure with the (pullback of the) map h. The homology of $S(F)$ vanishes in dimensions between 0 and r either by Leray spectral sequence or by constructing a nowhere 0 section of F and inducing a cell decomposition of $S(F)$ with cells of dimension 0, $r+1$, $n-r$ and $n+1$. The image of $H_{r+1}(S(F))$ is generated by $[h] = \alpha$, so the connected sum of W and $S(F)$ (with the normal structures defined above) does not change anything in dimensions under $r+1$ and makes $\pi_{r+1}(F)$ add α to its image.

Performing this step, we increase r to $r+1$, and we can do this until we reach $r = [n/2]$. The result is an $[n/2]-1$-smoothing and is bordant to the original normal structure relative to boundary.

3. (Andreas Angel)

First let us relate the second Stiefel-Whitney class to the second Wu class of the manifold.

From the equation $w(M) = Sq(v(M))$, we obtain that

$$w_k = \sum_{i+j=k} Sq^i(v_j)$$

and therefore $w_2(M) = v_2(M) + Sq^1 v_1(M) = v_2(M) + v_1(M)^2$.

But also $w_1(M) = v_1(M)$ and if the manifold is oriented (for example if it is simply connected) then $v_1(M) = 0$, and therefore,

$$w_2(M) = v_2(M)$$

Now let us use Poincare duality to relate the vanishing of the second Wu class with quadratic form on the second cohomology with coeficcients in $\mathbb{Z}/2$.

Note that if $x \in H^2(M; \mathbb{Z}/2)$ then

$$x \cup v_2(M) = x \cup x \quad \forall x \in H^2(M; \mathbb{Z}/2)$$

Since M is 4-dimensional, by Poincare duality (since M is closed, and every closed manifold is $\mathbb{Z}_2$-oriented),

$$v_2(M) = 0 \Leftrightarrow x \cup x = 0 \; \forall x \in H^2(M; \mathbb{Z}/2)$$

The coefficient sequence

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2 \to 0$$

induces a long exact sequence,

$$\cdots \to H^i(M; \mathbb{Z}) \to H^i(M; \mathbb{Z}) \to H^i(M; \mathbb{Z}/2) \xrightarrow{\beta} H^{i+1}(M; \mathbb{Z}) \to \cdots$$

If M has no 2-torsion in the first homology group (for example if it is simply connected) then by Poincare duality $H^3(M; \mathbb{Z})$ also has no 2-torsion and the Bockstein

homomorphism (the connecting homomorphism β) is zero. Therefore the reduction mod-2 map
$$H^2(M;\mathbb{Z}) \to H^2(M;\mathbb{Z}/2)$$
is surjective. And from this it follows that
$$\forall x \in H^2(M;\mathbb{Z}),\ x \cup x \text{ is even.}$$
if and only if
$$\forall x \in H^2(M;\mathbb{Z}_2),\ x \cup x = 0.$$
Since
$$v_2(M) = 0 \Leftrightarrow x \cup x = 0\ \forall x \in H^2(M;\mathbb{Z}/2)$$
and $w_1(M) = v_1(M)$, we are done.

4. (Haggai Tene)

Let M be a 1-connected 4-manifold with odd intersection form. We would like to show that the Gauss map $\nu : M \to BSO$ is a 2-equivalence which means that it is an isomorphism on π_0 and π_1 and a surjection on π_2. We start by computing π_i of BSO for $i \leq 2$. Note that there is a fibration $SO \to * \to BSO$ (where $*$ is a contractible space) which gives us a long exact sequence in homotopy groups:

$...\pi_k(SO) \to \pi_k(*) \to \pi_k(BSO) \to \pi_{k-1}(SO) \to \pi_{k-1}(*) \to \pi_{k-1}(BSO) \to ...$

We deduce that $\pi_k(BSO) \to \pi_{k-1}(SO)$ is an isomorphism for $k > 0$ so:

0) $\pi_0(BSO)$ is trivial.

1) $\pi_1(BSO) \cong \pi_0(SO)$ which is trivial

2) $\pi_2(BSO) \cong \pi_1(SO) \cong \mathbb{Z}/2$

Thus, to prove that ν is 2-equivalence we have to show that the map $\nu_* : \pi_2(M) \to \pi_2(BSO)$ is surjective or even non trivial since $\pi_2(BSO) \cong \mathbb{Z}/2$.

By exercise 3 above, the fact that S_M is odd implies that $\omega_2(M)$ is non zero. Using the fact that $\nu(M) \oplus TM$ is a trivial bundle we get by the Whitney formula that $\omega(M) \cup \omega(\nu(M)) = 1$ so $\omega_2(M) + \omega_2(\nu(M)) = 0$ (here we use the fact that M is simply connected so $H^1(M, \mathbb{Z}/2)$is trivial) so $\omega_2(\nu(M))$ is non zero. Since this element lies in the image of the map $H^2(BSO, \mathbb{Z}/2) \to H^2(M, \mathbb{Z}/2)$, so this map is non zero which implies that its dual map $H_2(M, \mathbb{Z}/2) \to H_2(BSO, \mathbb{Z}/2)$ is non trivial.

We look at the reduction mod 2 map:

$$\begin{array}{ccc} H_2(M) & \to & H_2(BSO) \\ \downarrow & & \downarrow \\ H_2(M, \mathbb{Z}/2) & \to & H_2(BSO, \mathbb{Z}/2) \end{array}$$

The map $H_2(M) \to H_2(M, \mathbb{Z}/2)$ is surjective by using the Bockstein sequence:

$... \to H_2(M) \xrightarrow{\times 2} H_2(M) \to H_2(M, \mathbb{Z}/2) \to H_1(M) \to ...$

and the fact that M is 1-connected so $H_1(M)$ is trivial. Therefore, the map $H_2(M) \to H_2(BSO)$ is non zero.

Look at the following square:

$$\begin{array}{ccc} \pi_2(M) & \to & \pi_2(BSO) \\ \downarrow & & \downarrow \\ H_2(M) & \to & H_2(BSO) \end{array}$$

Since both spaces are 1-connected the vertical maps are isomorphisms, so the map $\pi_2(M) \to \pi_2(BSO)$ is non zero which finishes the proof.

Hint: recall $H^*(BSO; \mathbb{Z}/2) \cong \mathbb{Z}/2[w_2, w_3, w_4, \dots]$.

A1 We have the exact sequence:

$$0 \to L \xrightarrow{i} A \xrightarrow{b'} L^* \to 0$$

Where i is the inclusion and b' is the restriction to L of an isomorphism $x \mapsto b(x, \_)$ between A and A^*. It means that A is isomorphic to $L \oplus L^*$ (because L is a direct summand) and we can choose a basis of $L \oplus L^* \cong A$ so that the matrix of the form b takes form

$$\begin{bmatrix} 0 & I \\ I & B \end{bmatrix}$$

where B is a symmetric matric with integer entries and pair diagonal entries (b is even). Let $v_1, \dots, v_k$ be the basis of L and $w_i, \dots, w_k$ be the basis of L^*. Let $1 \le j \le k$. We are going to construct the vectors $w'_1, \dots, w'_j$ such that $span\{w'_1, \dots, w'_j\} = span\{w_1, \dots, w_j\}$ and the matrix of b in the basis $v_1, \dots, v_k, w'_1, \dots, w'_j, w_{j+1}, \dots w_k$ is of the form

$$\begin{bmatrix} 0 & I \\ I & B' \end{bmatrix} \quad \text{where } B' = \begin{bmatrix} 0 & X \\ Y & Z \end{bmatrix}$$

If $j = k$ then we are done. Suppose we have found w'_i like this for all $0 \le i < j$. We want to add w'_j to this list so that all conditions stated above are verified.
Let $b(w_j, w_j) = 2a$, $b(w_j, w'_i) = a_i$ for $0 \le i < j$. We put

$$w'_j = -av_j + w_j + \sum_{i=1}^{i=j-1} -a_i v_i$$

Obviously we don't change the span of the respective vectors. Let's verify the conditions put on the matrix:

(a) $b(w'_j, v_i) = 0$ for $i < j$,
(b) $b(w'_j, v_j) = 1$,
(c) $b(w'_j, v_i) = 0$ for $i > j$,
(d) $b(w'_j, w'_i) = a_i - a_i = 0$ for $i < j$,
(e) $b(w'_j, w'_j) = -2a + 2a = 0$.

By inductive step, the proof is completed.

A2 ...

6 Friday

1. (Jan Czajkowski)

   We have
   $$\omega(\alpha) = (S^{p+q+1} \times [0;1]) \cup_{f_\alpha} (D^{p+1} \times D^{q+1})$$
   and
   $$\chi(\alpha) = \partial_+(\omega(\alpha)).$$
   We want to express $\chi(\alpha)$ and $\omega(\alpha)$ in terms of $[\alpha] \in \pi_p(\mathrm{SO}_{q+1})$.

   Remark 7. All non-smooth manifolds (like e. g. $D^{p+1} \times D^{q+1}$) are considered smoothed in some standard way.

   I calculate $\omega(\alpha)$:

   $$\begin{aligned}\omega(\alpha) = &\underbrace{(S^{p+q+1} \times [0;1])}_{\cong D^{p+1}\times D^{q+1}\setminus \mathrm{int}\left(\frac{1}{2}(D^{p+1}\times D^{q+1})\right)} \cup_{f_\alpha} (D^{p+1} \times D^{q+1}) \\ &\cong (D^{p+1} \times D^{q+1}) \cup_{f_\alpha} (D^{p+1} \times D^{q+1}) \setminus \mathrm{int}\left(\tfrac{1}{2}(D^{p+1} \times D^{q+1})\right) \qquad (2)\end{aligned}$$

   (because $\mathrm{int}\left(\frac{1}{2}(D^{p+1} \times D^{q+1})\right)$ is disjoint with the gluing area)

   $$\cong \mathcal{B}(D^{p+1} \times \mathbb{R}^{q+1}) \cup_{f_\alpha} \mathcal{B}(D^{p+1} \times \mathbb{R}^{q+1}) \setminus \mathrm{int}\left(\tfrac{1}{2}(D^{p+1} \times D^{q+1})\right)$$

   (where $\mathcal{B}(\cdot)$ denotes the balls bundle of a vector bundle)

   $$\cong \mathcal{B}(D^{p+1} \times \mathbb{R}^{q+1} \cup_{f_\alpha} D^{p+1} \times \mathbb{R}^{q+1}) \setminus \mathrm{int}\left(\tfrac{1}{2}(D^{p+1} \times D^{q+1})\right)$$

   (here f_α is extended to $S^p \times \mathbb{R}^{q+1}$)

   $$\cong \mathcal{B}(\mathbb{R}^{q+1}\text{-vector bundle over } S^{p+1} \text{ induced by } [\alpha] \in \pi_p(\mathrm{SO}_{q+1})) \setminus \text{an open ball.}$$

   Now, let us calculate $\chi(\alpha)$:

   $$\begin{aligned}\chi(\alpha) &= \partial_+(\omega(\alpha)) = \partial(\omega(\alpha)) \setminus \partial(\underbrace{\tfrac{1}{2}(D^{p+1} \times D^{q+1})}_{\text{the part substracted in (2)}}) &(3)\\ &\cong \partial((D^{p+1} \times D^{q+1}) \cup_{f_\alpha} (D^{p+1} \times D^{q+1})) &(4)\\ &\cong \partial\mathcal{B}(\mathbb{R}^{q+1}\text{-vector bundle over } S^{p+1} \text{ induced by } [\alpha] \in \pi_p(\mathrm{SO}_{q+1})) &(5)\\ &\cong \mathcal{S}(\mathbb{R}^{q+1}\text{-vector bundle over } S^{p+1} \text{ induced by } [\alpha] \in \pi_p(\mathrm{SO}_{q+1})), &(6)\end{aligned}$$

   the spheres bundle of underlying vector bundle.

2. (Jan Czajkowski)

   Let us notice that $S^{p+q+1} = S^p \times D^{q+1} \cup_{S^p\times S^q} D^{p+1} \times S^q$. I calculate $\chi(\alpha)$ in another way than in ex. 1.1.:

$$\chi(\alpha) = (\underbrace{S^{p+q+1} \times \{0\}}_{\partial_+(S^{p+q+1}\times[0;1])}) \cup_{f_\alpha} S^{p+q+1} \setminus \mathrm{int}(\underbrace{S^p \times D^{q+1}/f_\alpha}_{(\mathrm{dom} f_\alpha)/f_\alpha\text{, the glued part}}) \tag{7}$$

$$\cong (S^p \times D^{q+1} \cup_{S^p\times S^q} D^{p+1} \times S^q) \cup_{f_\alpha} (S^p \times D^{q+1} \cup_{S^p\times S^q} D^{p+1} \times S^q) \setminus \mathrm{int}(S^p \times D^{q+1}/f_\alpha) \tag{8}$$

$$\cong \underbrace{D^{p+1} \times S^q}_{=\mathcal{S}(D^{p+1}\times\mathbb{R}^{q+1})} \cup_{f_\alpha\restriction S^p\times S^q} \underbrace{D^{p+1} \times S^q}_{=\mathcal{S}(D^{p+1}\times\mathbb{R}^{q+1})} \cong \mathcal{S}(D^{p+1} \times \mathbb{R}^{q+1} \cup_{f_\alpha\restriction S^p\times S^q} D^{p+1} \times \mathbb{R}^{q+1}) \tag{9}$$

$$\cong \mathcal{S}(\mathbb{R}^{q+1}\text{-vector bundle over } S^{p+1} \text{ induced by } [\alpha] \in \pi_p(\mathrm{SO}_{q+1})). \tag{10}$$

   Hence $\chi(\alpha)$ consists of two "spheres" (precisely: copies of $D^{p+1} \times S^q$): the "sphere"

$$D^{p+1} \times S^q \subseteq S^{p+q+1} \subseteq D^{p+1} \times D^{q+1}$$

   and the "dual sphere"

$$D^{p+1} \times S^q \subseteq \partial_+(S^{p+q+1} \times [0;1]) = S^{p+q+1} \times \{0\}.$$

3. ...

A1 ...

A2 (Andreas Angel)

   Note that $W(\mathbb{Z})$ is the stable isomorphism classes of unimodular bilinear forms, where we stabilize with symmetric forms that contain a lagrangian (metabolic forms).

   The signature is additive with respect to the direct sum and for a bilinear form with a Lagrangian, we know from a previous exercise that the signature is zero, therefore it gives a well defined homomorphism

$$W(\mathbb{Z}) \to \mathbb{Z}$$

   To see that this homomorphism is an isomorphism, we need a classification result about indefinite forms of odd type.

   Lemma 8. *A bilinear symmetric indefinite unimodular form of odd type has an orthonormal basis, therefore it is the direct sum of copies of* $[1]$ *and* $[-1]$.

By induction. We will need the following deep fact:

Every unimodular symmetric form b that is indefinite contains a non-zero vector x, such that $b(x,x)=0$

Choose a vector $x_1 \neq 0$ such that $b(x_1,x_1)=0$. We can suppose that x_1 extends to a basis $x_1,\cdots,x_n$, by unimodularity find $y_1,\cdots,y_n$ that corresponds to a dual basis. In particular $b(x_1,y_1)=1$.

Since the form b is odd, there exists y such that $b(y,y)$ is odd. Therefore there exists y_i such that $b(y_i,y_i)$ is odd.

If $b(y_1,y_1)$ is odd consider the sub-lattice $A_0=\mathrm{Span}(x_1,y_1)$ and if $b(y_1,y_1)$ is even, then consider $A_0=\mathrm{Span}(x_1,y_1+y_i)$. Which in any case is a rank 2 lattice.

In both cases, with respect to that basis $A_0=\mathrm{Span}(X,Y)$ (to give some names), the form is represented by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix}$$

where a is odd. If $a=2k+1$, then consider the vectors

$$X'=Y-kX \text{ and } Y'=Y-(k+1)X$$

With respect to this new basis of A_0, the form is represented by

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Since restricted to A_0 the form is unimodular, then

$$A \cong A_0 \oplus A_0^{\perp}$$

i.e.

$$b \cong [1] \oplus [-1] \oplus b|_{A_0^{\perp}}$$

By choosing the sign correctly

$$[\pm 1] \oplus b|_{A_0^{\perp}}$$

is a symmetric unimodular indefinite form of odd type and of lower rank, and by induction it is a direct sum of copies of [1] and [−1]. □

This signature homomorphism is surjective since the signature of the form [1] is 1. Now if the signature of a form b is zero then the sum $b \oplus [1] \oplus [-1]$ is indefinite and of odd type. Since the signature of b is zero, the signature of $b \oplus [1] \oplus [-1]$ is also zero

By the previous lemma $b \oplus [1] \oplus [-1]$ is isomorphic to a copy of forms [1] and [−1] and there should be the same number of 1 and −1 because the signature is zero. Such form represents the zero element in $W(\mathbb{Z})$, because clearly it contains a Lagrangian. In symbols,

$$b \sim b \oplus [1] \oplus [-1] \cong \bigoplus [1] \oplus [-1] \sim 0$$

where $\sim$ means stably isomorphic (by forms containing a lagrangian).