The stable classification of 4-manifolds
Notes from the lectures of Matthias Kreck
Krakow, August 2010

These notes were TeXed by Anna Abczynski, Lukasz Bak and Andrzej Czarnecki who attended the Krakow summer school on the stable classification of 4-manifolds given by Matthias Kreck in Krakow, August 2010. The notes are in the process of being revised and edited by Diarmuid Crowley.

Contents

1 Motivational problems and questions

An important and long standing problem in topology is to understand and classify n-dimensional closed manifolds. In dimension 2 there has been a classification since 18?? []. The situation in dimension 4 is extremely complex. But that is exactly where we are going to tackle it.

Let's begin by listing some 4-manifolds.

- $\mathbb{R}^4$
- $\mathbb{S}^4$
- $\mathbb{C}P^2$
- T^4
- $\Sigma_g \times \Sigma_{g'}$, products of two orientable surfaces and connected sums of those
- Kummer surface, or - more generally - for every $k \in \mathbb{Z}$ the space obtained as $\{x \in \mathbb{C}P^3 | \sum x_i^k = 0\}$
- $\mathbb{S}^1 \times S^3$
- $\mathbb{C}P^2/\sim$, where $\sim$ is complex conjugation or

- $\mathbb{S}^1 \times$ Heisenberg manifold.

When are these manifolds homeomophic, diffeomorphic or homotopy equivalent? Observe: all manifolds in the list (apart from V_k) can be distinguished by their homology groups.

As for the concrete example, we will mostly consider the manifolds

$$V_4\#(-\mathbb{C}P^2) \quad \text{and} \quad \#_a\mathbb{C}P^2\#_b(-\mathbb{C}P^2).$$

2 Homotopy classification of 1-connected closed 4-manifolds

We aim for the result of Whitehead and Milnor. We shall start with some homotopy-theoretic preliminaries.

1. Every manifold is homotopy equivalent to some CW-complex. If the manifold is compact, so is the CW-complex, hence finitness of homology. Compare [2], Corollary E.5. in Appendix E.

2. Whitehead's Theorem. For CW-complexes X and Y, the following conditions are equivalent:

- $f : X \to Y$ is a homotopy equivalence,
- $f_* : \pi_k(X) \to \pi_k(Y)$ are isomorphisms for all k,
- $f_* : \pi_1(X) \to \pi_1(Y)$ is an isomorphism and $\widetilde{f}_* : H_k(\widetilde{X}) \to H_k(\widetilde{Y})$ are isomorphisms for all k where $\widetilde{X}$ and $\widetilde{Y}$ are the universal coverings and $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ is the induced map. Compare [17].

Thus to show that 1-connected compact 4-manifolds X and Y are homotopy equivalent we need only find a map $f : X \to Y$ such that $f_* : H_2(X) \to H_2(Y)$ and $f_* : H_4(X) \to H_4(Y)$ are isomorphisms, since by Poincaré Duality, Universal Coefficients Theorem and Hurewicz Theorem we have

$$H_3(X) \cong H^1(X) \cong \mathrm{Free}(H_1(X)) \oplus \mathrm{Tor}(H_0(X)) \cong H_1(X) \cong \pi_1(X) = 0.$$ [1]

[1] We will naturally assume that homology/homotopy groups of X and Y are isomorphic, since otherwise they can't be homotopy equivalent.

Obviously, we lack a candidate for such a map. We start by comparing our manifolds with some other space. Recal that for each ring with unity R there is a one to one correspondence

3. $H_i(X;R) \cong [X, K(R,i)]$

between X's i-th cohomology with R coefficients and homotopy classes of maps from X to Eilenberg-MacLane space $K(R,i)$, universal object defined (up to homotopy equivalenece) by it's unique nontrivial homotopy group $\pi_i(K(R,i)) \cong R$, compare [16]. We want to use it for $H_2(X;\mathbb{Z})$, obtaining for each cohomology class a map to $\mathbb{C}P^\infty$, the infinite complex projective space[2]. By Poincaré Duality and Universal Coefficients Theorem we get

$$H_2(X) \cong H^2(X) \cong \mathrm{Free}H_2(X) \oplus \mathrm{Tor}H_1(X).$$

Since the latter summand is zero, second homology is free and we can pick it's basis $a_1, \ldots, a_r$[3]. Each class corresponds to some map into $\mathbb{C}P^\infty$, and collection of these gives us a map f_X from X to $K \overset{\text{def.}}{=} \prod_{1,\ldots,r} \mathbb{C}P^\infty$. Note that (by Künneth) it induces an isomorphism in second homology. Applying the same to Y, we get a diagram

$$\begin{array}{ccc} & & Y \\ & & \downarrow \\ X & \longrightarrow & K \end{array}$$

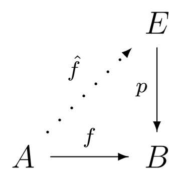
which we want to close with an arrow $g : X \to Y$, such that the triangle will commute up to homotopy. Then g will induce a honest isomorphism in second - and hopefully also in the top homology. Looking for such g, we are invited to explore the obstruction theory.

[2]This correspondence is as follows: each class α is given the unique (up to homotopy) map f_α, such that $\alpha = f_\alpha^*(\iota)$, where ι is single generator of $H^2(\mathbb{C}P^\infty) = \mathbb{Z}$.

[3]Since we will use this argument again and again as a root of all "free-group" claims, from now on we will refrain from evoking it every time.

3 Obstruction theory.

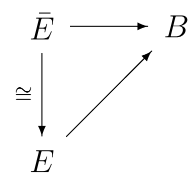
Consider a fibration $p : F \hookrightarrow E \to B$ and a map from A to B. We are considering problem of closing the diagram[4]



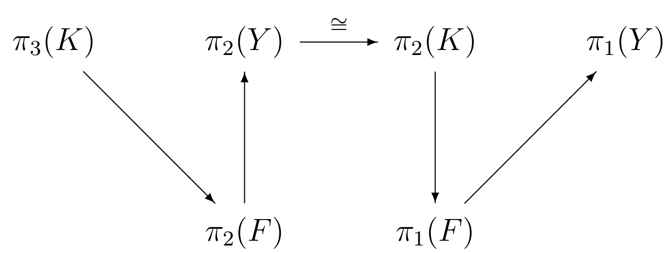
4. There exists a sequence of cohomology classes $\Theta_i \in H^i(B; \pi_{i-1}(F))$ such that $\Theta_i = 0$ if and only if the lift $\hat{f}$ exists.

We don't need the details, yet they are to be find in [16]. We just need to adjust our setting.

5. For every map $\bar{E} \to B$ there exist homotopy equivalence and fibration $E \to B$ fiting into diagram



Therefore we can - ad will - assume that $f_Y : Y \to K$ is a fibration with fiber F and consider obstructions to lift f_X. We don't have explicit form of F, but we know that it fits into long exact sequence of fibration[5]:



The horizontal map is an isomorphism by Hurewicz Theorem, and outmost groups are trivial, hence also the bottom groups. Thus we dispatched obstructions Θ_2 and Θ_3 and by 1-connectedness of X we are only left with Θ_4 to deal with. We will use the folowing fact

[4]Differently speaking: lifting the map f to E.

[5]Again, details to be found in [16].

6. For every map $\bar{E} \to B$ there exist homotopy equivalence and inclusion fiting into diagram

$$\begin{array}{ccc} \bar{E} & \longrightarrow & B \\ & \searrow^{\subset} & \uparrow \cong \\ & & CE \end{array}$$

The space CE can be realized by a maping cylinder over considered map.

So we can - an will - assume that f_Y is an inclusion[6] and consider the homotopy long exact sequence of the pair (K, Y)

$$\begin{array}{ccccccc} 0 & \longrightarrow & \pi_4(K,Y) & \longrightarrow & \pi_3(Y) & \longrightarrow & \pi_3(K) \\ & & \downarrow & & \downarrow = & & \downarrow = \\ 0 & \longrightarrow & \pi_3(F) & \longrightarrow & \pi_3(Y) & \longrightarrow & \pi_3(K) \end{array}$$

compared with the long exact sequence of a fibration. One easily checks that middle arrow is an isomorphism (for example using Five Lemma). Therefore we can harmlessly change coefficients and consider our obstruction as an element $\Theta_4 \in H^4(X, \pi_4(K,Y)) \cong \mathrm{Hom}(H_4(X), \pi_4(K,Y))$. Since we can assume that map from 3-skeleton of X, $g : X^3 \to Y \subset K$ is already constructed, it turns out that Θ_4 is composition:

$$H_4(X) \longrightarrow H_4(X, X^3) \xrightarrow{(f_X)_*} H_4(K,Y) \xrightarrow{\cong} \pi_4(K,Y)$$

the last isomorphism due to (relative) Hurewicz Theorem. Now we know that Θ_4 vanishes if and only if $(f_X)_*[X] = 0$ in the fourth relative homology, so we can use exact sequence of the pair

$$\langle [Y] \rangle \cong H_4(Y) \to H_4(K) \to H_4(K,Y) \to 0$$

to write $H_4(K,Y) \cong H_4(K)/H_4(Y)$ and conclude that

$$\Theta_4 = 0 \iff (f_X)_*[X] = 0 \in H_4(K)/_{\langle (f_Y)_*[Y] \rangle}$$

[6] We just need to be careful not to assume inclusion and fibration at the same time.

The next step is to get rid of K. By Poincaré Duality argument and with f_X and f_Y inducing isomorphisms on second cohomology, one can show that the last condition is equivalent to the next one, and then elaborate:[7]

$$\begin{aligned}
& (f_X)_*[X] = (f_Y)_*[Y] \\
\iff\ & \forall \alpha \in H^4(K) : \langle \alpha, (f_X)_*[X] \rangle = \langle \alpha, (f_Y)_*[Y] \rangle \\
\iff\ & \forall \alpha, \beta \in H^2(K) : \langle \alpha \smile \beta, (f_X)_*[X] \rangle = \langle \alpha \smile \beta, (f_Y)_*[Y] \rangle \\
\iff\ & \forall \gamma, \delta \in H^2(Y) : \langle f_X^*((f_Y)^*)^{-1}\gamma \smile f_X^*((f_Y)^*)^{-1}\delta, [X] \rangle \\
& = \langle \gamma \smile \delta, [Y] \rangle.
\end{aligned}$$

Thus we arrived to the conclusion:

Proposition 3.1. Map g exists if and only if there is an isomorphism $\rho\colon H^2(Y) \to H^2(X)$ preserving cup product, that is for all $\gamma, \delta \in H^2(Y)$ we have

$$\langle \rho(\gamma) \smile \rho(\delta), [X] \rangle = \langle \gamma \smile \delta, [Y] \rangle.$$

What remains, is the control of the degree, behavior of g^* in the top cohomology. We owe to Poincaré Duality that cup pairing in second cohomology is not trivial so we can split the fundamental cohomology class $[X] = \gamma \smile \delta$ by second cohomology classes and go through $f_X^*(f_Y^*)^{-1}$, which we know to be an isomorphism in second cohomology. As we have nowhere used that the spaces under consideration are manifolds, we obtain

Theorem 3.2. Whitehead-Milnor. Two 1-connected compact 4-dimensional CW-complexes X and Y, exhibiting Poincaré Duality are homotopy equivalent if and only if there is an isomorphism $\rho : H^2(Y) \to H^2(X)$ which preserves the cup product.

4 The Intersection form of a $2k$-manifold

7. The *intersection form* on a closed oriented $4k$-dimensional manifold M is defined to be:

$$\mathrm{Free}(H^{2k}(M)) \otimes \mathrm{Free}(H^{2k}(M)) \to \mathbb{Z}$$

$$(\alpha, \beta) \mapsto S_M(\alpha, \beta) \overset{\text{def.}}{=} \langle \alpha \smile \beta, [M] \rangle$$

[7]Here, angle brackets standing for evaluation of cohomology class on homology class.

This is an unimodular symmetric bilinear form on a finitely generated free abelian group, so we may write: $S_M = (b(a_i, a_j))_{i,j}$ in some basis. If M is 1-connected and 4-dimensional ($k = 2$), it contains all information about the cup product, so we can rephrase our previous work:

8. Classification of 4 manifolds up to homotopy equivalence is an arithmetic problem of classification of symmetric, bilinear forms over $\mathbb{Z}$, $(\mathbb{Z}^r, S_M)$ up to form-preserving isomorphisms.

It turns out to be rather hard. Consult Serre [11] or Milnor-Husemoller [7] to learn about the easier parts of this problem.

First, we need to figure out what kind of invariants we want to consider. The simplest is the rank of $\mathbb{Z}^r$. More enlightening is to consider the signature of the matrix, that is the number of positive minus the number of negative eigenvalues of diagonalisation over $\mathbb{R}$ (ocasionaly we regard S_M as a matrix over $\mathbb{R}$, but bear in mind that we are interested in calssifying matrices over $\mathbb{Z}$). We will write $\operatorname{sign} M$ for $\operatorname{sign} S_M$. Signature classifies all bilinear forms over $\mathbb{R}$, but over $\mathbb{Z}$ we also need the so-called type. We say that S_M is of even type if $S_M(x, x)$ is even for all $x \in \mathbb{Z}^r$ and of odd type otherwise.

9. [8] Two indefinite unimodular symmetric forms over $\mathbb{Z}$ are equivalent if and only if all these three invariants agree.

For a moment we will be interested in even positive definite forms.

10. Van der Blij, [14]. Even positive definite forms exist only if

$$\operatorname{rank} \mathbb{Z}^r = 0 \mod 8.$$

The picture is as follows.

rank	8	16	24	32
# of unimodular positive definite forms of even type	1	2	24	> 80000000
given by	E_8	$E_8 \oplus E_8$, E_{16}	f.e. Leech lattice	???

We may ask: are the intersection forms of the manifolds listed in the beginning definite? The answer is positive only for $\mathbb{C}P^2$ and connected sums of $\mathbb{C}P^2$. We will check it - and exploit it - in case of V_k. Easy criterion for

[8] Reference needed

positive definite matrices is $\operatorname{sign} S_M = \operatorname{rank} \mathbb{Z}^r$. Hence the question what is the signature of V_k and what is the rank $H^2(V_k)$ or - equivalently - $\chi(V_k)$, the Euler characteristic (for 1-connected compact 4-manifolds the condition $\operatorname{rank} H^2(M) = \chi_M - 2$ is satisfied).

The following will highlight importance of characteristic classes, which we develop in following chapters. We will use the complex structure on V_k and compute Chern classes $c_i(TV_k) = c_i(V_k) \in H^{2i}(V_k)$.

11. For any compact complex manifold M of complex dimension k we have $\langle c_k(M), [M]\rangle = \chi(X)$.

It will be convenient for us to find that $\langle c_2(V_4), [V_4]\rangle = e(V_4) = 24$.

There is another beautiful theorem of this type, stating:

12. For compact complex surface ($\dim_{\mathbb{C}} M = 2$),

$$\operatorname{sign} M = \frac{1}{3}\langle c_1(M) \cup c_1(M) - 2c_2(M), [M]\rangle.$$

In particular, we will find that $\operatorname{sign} V_4 = \frac{1}{3}(0 - 48) = -16$ and so the intersection form of V_4 is indefinite.

Now consider our example $V_4\#(-\mathbb{C}P^2)$. By Mayer-Veitrois argument, the signature is additive under connected sum, so we get $\operatorname{sign} V_4\#(-\mathbb{C}P^2) = -17$, and $\chi(V_4\#(-\mathbb{C}P^2)) = 25$. On the other hand we had $\#_a\mathbb{C}P^2\#_b(-\mathbb{C}P^2)$ with $\operatorname{sign} \#_a\mathbb{C}P^2\#_b(-\mathbb{C}P^2) = a - b$ and $\chi(\#_a\mathbb{C}P^2\#_b(-\mathbb{C}P^2)) = a + b + 2$. Comparing all relevant invariants, we wrap it up in

13. $V_4\#(-\mathbb{C}P^2)$ is homotopy equivalent to $\#_a\mathbb{C}P^2\#_b(-\mathbb{C}P^2)$ if and only if $a = 3$ and $b = 20$.

It gets better:

14. Freedman '82, [4]. They are also homeomorphic.

But only to some extent:

15. Donaldson '84, [3]. [9] They are not diffeomorphic.

Freedman's theorem covers the part of Milnor-Whitehead classification that we ignored: we know each 1-connected closed 4-manifold is homotopically distinguished by it's intersection form, but we didn't try to answer which unimodular symetric matrices are actually realized as an intersection form of some topological 4-manifold. It turns out, that each one is.

[9] Probably not the best reference, and three years later than the theorem itself.

16. FREEDMAN'S CLASSIFICATION OF CLOSED SIMPLY-CONNECTED 4-MANIFOLDS. Classes of homeomorphism of 4-manifolds are in one-to-one correspondence with pairs (S, ks) where S is an unimodular symmetric form, $ks \in \mathbb{Z}_2$ is called a Kirby-Siebenmann obstruction and if S is even, then $\frac{\text{sign}(S)}{8} = ks \pmod 2$.

There is no flavour of differentiability in this theorem. However we will mention - as a curiosity - that 30 years before Vladimir Rokhlin proved (in [10], but compare [8]) the following

17. Signature of intersection form of smooth 1-connected 4-manifold is divisible by 16.

And so by Freedman's result, we know that there is a 4-manifold realizing E_8 as it's intersection form, hence it is not homeomorphic to any smooth manifold.

5 Bordism.

If we hope for a homeomorphism classification of 4-manifolds, we are disappointed by a by-product of the "Word problem". Given two finite sets of generators and two sets of relations for them, one cannot determine, if they represent the same group. But for each finitely presented group one can easily construct a 4-manifold with fundamental group having the same presentation, therefore algorithm to distinguish 4-manifolds would distinguish finitely presented groups. A little less rigid classification (but more bounded to manifold category than homotopy type) is by the bordism relation, introduced by René Thom around 1950[10].

18. M is *bordant* to N if there exists a compact manifold W with boundary $\partial W = M \sqcup -N$, where minus denotes change of orientation. We write $M \simeq N$ and define

$$\Omega_k \stackrel{\text{def.}}{=} \{M | M \text{ closed oriented of dimension } k\}/_{\simeq}$$

This is a group under the disjoint sum or - equivalently - connected sum (one easily checks that $(M \sqcup N) \simeq (M \# N)$). Moreover,

$$\Omega_* \stackrel{\text{def.}}{=} \bigoplus_k \Omega_k$$

[10] Although general concept may be tracked down as far as in Riemann's work.

is a ring with multiplication given by cartesian product (although one has to put some effort to remain in smooth category). The fundamental results for bordism theory are characterisation of this ring:

19. THOM. $\Omega_* \otimes \mathbb{Q} \cong \mathbb{Q}[\mathbb{C}P^2, \mathbb{C}P^4, ...]$

and connection between bordism and homotopy type:

20. ROKHLIN. $M^{4k} \mapsto \text{sign}(M)$ induces a well defined group epimorphism $\Omega_{4k} \to \mathbb{Z}$. In particular: signature is a bordism invariant, $\text{sign}\,\mathbb{C}P^2 = 1$ and for W^{4k+1} compact and oriented $\text{sign}\,\partial W = 0$, thus two $4-$manifolds are bordant if and only if they have the same signature.

both to be found in [13] [11]

6 Characteristic classes made simple.

We now develop characteristic classes in scope needed for this lectures. Restrict atention only to vector bundles $p : E^k \to M^m$ of rank k over closed m-manifolds, each object smooth and oriented[12]. Our goal is to construct a cohomology class $e(E) \in H^k(M)$, the Euler class. Let $s : M \to E$ be the zero section and deform s to s', transverse to s, so that

$$V = \{x \in M | s'(x) = s(x) = 0\}$$

be an $m - k$ dimensional submanifold of M. The Euler class arises as composition

21. Define

$$e(E) \overset{\text{def.}}{=} PD^{-1}(i_*[V]) \in H^k(M)$$

with the following properties:

- $e(E \times E') = e(E) \times e(E') \in H^{M \times M'}$ where $E \times E'$ is a bundle over $M \times M'$.
- $f\colon M \to N$ then $e(f^*E) = f^*e(E)$.
- $e(E \oplus E') = e(E) \cup e(E')$

[11] I don't know, if it is the best reference, because of the language barrier.

[12] Oriented vector bundles are exactly these that can be reduced to the group SOk

- Let $\mathbb{C} \hookrightarrow L \to \mathbb{C}P^n$ be the canonical line bundle[13]. Then

$$e(L) = D^{-1}(i_*[\mathbb{C}P^{n-1}]) = x,$$

where x generates $H^2(\mathbb{C}P^n)$.

Let now $E^k \to M^m$ be a complex vector bundle of rank k. Consider $E^k \otimes L \to M^m \times \mathbb{C}P^N$ for some big N, a complex vector bundle of the same rank. Thanks to K unneth formula we can write down the Euler class of this bundle, happily recognizing Chern classes

22. $H^{2k}(M \times \mathbb{C}P^N) = H^{2k}(M) \oplus H^{2k-2}(M) \oplus \ldots \oplus H^2(M) \oplus H^0(M)$,

$$c(E) \overset{\text{def.}}{=} e(E \otimes L) = c_k(E) + c_{k-1}(E) + \ldots c_1(E) + c_0(E)$$

inheriting properties from the Euler class:

- $c(E \oplus F) = c(E) \smile c(F)$
- $c(f^*E) = f^*c(E)$
- $c(L) = 1 + x$.

To get characteristic classes for real bundles we proceed with the following trick. Given a real vector bundle $E^k \to M^m$ we complexify it and define Pontryagin classes

23. $p_i(E) \overset{\text{def.}}{=} (-1)^i c_{2i}(E \otimes_{\mathbb{R}} \mathbb{C})$ behaving similarly.

Finally we want to define the Thom class. This time equip $E^k \to M^m$ with a Riemannian metric and consider the disc bundle DE. By means of zero section define

24. $u(E) \overset{\text{def.}}{=} s_*[M] \in H^k(DE, SE) \cong H_m(DE)$

last isomorphism given by Lefschetz duality.

25. $H^r(M) \to H^{r+k}(DE, SE)$ is an isomorphism sending $\alpha \mapsto p^*(\alpha) \smile u(E)$.

[13] Simplest way to put it, a fiber over a point in projective space is exactly the subspace in $\mathbb{C}^{n+1}$ represented by this point.

7 Application to bordism.

We will use characteristic classes to reformulate the Thom Theorem. Take any partition of $m = i_1 + \ldots + i_r$ and assign to M^{4m} it's $i_1, \ldots, i_r$-th Pontryagin number $\langle p_{i_1} \smile \ldots \smile p_{i_q}, [M^{4k}]\rangle$. Observe that for a boundary $\partial W = N \hookrightarrow W$ we get:

$$\begin{aligned}\langle p_{i_1}(N) \smile \ldots \smile p_{i_r}(N)), [N]\rangle &= \langle i^* p_{i_1}(W) \smile \ldots \smile p_{i_r}(W), [M]\rangle \\ &= \langle p_{i_1}(W) \smile \ldots \smile p_{i_r}(W), i_*[M]\rangle = 0,\end{aligned}$$

equalities following from naturality of Pontryagin classes and triviality of the normal bundle to N, and concluding zero from the fact that boundary is homologous to zero in ambient manifold. This shows that for each partition of k we have a well defined function over bordism classes. Moreover (references as before)

26. Thom. $\Omega_{4k} \otimes \mathbb{Q} \to \mathbb{Q}^{\#\text{number of partitions of } k}$ is an isomorphism.

which is reformulation of Thom's theorem.

We begin with announced computation of signature for 4-manifolds.

27. Rokhlin. For a closed oriented 4-manifold M, $\operatorname{sign} S_M = \frac{1}{3}\langle c_1(TX) \smile c_1(TX) - 2c_2(TX), [X]\rangle = \frac{1}{3}\langle p_1(M), [M]\rangle$.

Proof. Equality between the cohomology classes follow from definition of Pontryagin classes. By invariance of signature over bordism, behaviour of Pontryagin classes and the fact that $\Omega_4 \otimes \mathbb{Q} \cong \mathbb{Q}[\mathbb{C}P^2]$, the formula is true if and only if $\operatorname{sign} S_M = \frac{1}{3}\langle p_1(\mathbb{C}P^2), [\mathbb{C}P^2]\rangle$ This is actualy a bit tricky, if we only have Thom theorem "over $\mathbb{Q}$", can we conclude this formula? it of course follows from the fact that $\Omega_4 \cong \mathbb{Z}[\mathbb{C}P^2]$. Therefore we need to compute characteristic classes of $\mathbb{C}P^2$.

28. $T\mathbb{C}P^n \oplus \mathbb{C} \cong (n+1)(\overline{L})$

Thus we know $p_1(\mathbb{C}P^2) = p_1(3\overline{L}) = 3p_1(\overline{L})$ and by definition

$$p_1(L) = -c_2(L \otimes \mathbb{C}) = -c_2(L \oplus \overline{L}) = -c_1(L) \smile c_1(\overline{L})$$

Thus $\langle p_1(\mathbb{C}P^2), [\mathbb{C}P^2]\rangle = \langle 3x^2, [\mathbb{C}P^2]\rangle = 3$ which is exactly three times the signature of $\mathbb{C}P^2$. □

We can now return to our initial example: distinguish homotopy type of $V_k\#(-\mathbb{C}P^2)$, so we compute Chern classes of the first summand. Naturally $i : V_k \hookrightarrow \mathbb{C}P^3$ and we have $T\mathbb{C}P^3|_{V_k} = TV_k \oplus \nu(i)$, last term being the normal bundle to this inclusion. In terms of first Chern class, the left hand side reads $c_1(TV_k) + c_1(\nu(i))$, the latter being equal to ki^*x (x coming from natural inclusion of $\mathbb{C}P^2$ into $\mathbb{C}P^3$ as a generator of the second cohomology). Since the right hand side is $i^*c_1(\mathbb{C}P^3) = i^*c_1(4\overline{L}) = 4i^*x$, we get

29. $c_1(V_k) = (4 - k)i^*x$.

We treat second Chern class similarly. The splitting of the bundle gives us - on the right - $c_2(V_k) + c_1(V_k) \smile c_1(\nu(i))$ and so from the preceding and $c_2(\mathbb{C}P^3) = 6i^*x^2$ we get

30. $c_2(V_k) = (6 - k(4 - k))i^*x^2$.

Evaluation of appriopriate classes establishes the figures given before and closes the discussion of this example.

8 Digression: complete intersections.

In the considerations above we tacitly used that second cohomology behaves nicely between $\mathbb{C}P^2$, $\mathbb{C}P^3$ and V_k. There is a general fact:

31. Lefschetz hyperplane theorem. Consider r homogeneous complex polynomials of various degrees in $n + r + 1$ variables, $f_1, \dots, f_r$. We can define $V_{f_1,\dots,f_r} \overset{\text{def.}}{=} \{x \in \mathbb{C}P^{n+r} \colon f_i(x) = 0 \text{ for all } i\} \subset \mathbb{C}P^{n+r}$. We call this space a *complete intersection*. Suppose 0 is a "regular" value. Then $V_{f_1,\dots,f_r}$ is a smooth submanifold and the inclusion $i : V_{f_1,\dots,f_r} \to \mathbb{C}P^{n+r}$ is an n-equivalence[14].

There is a very interesting problem of classifying these spaces

32. Sullivan conjecture. Two such V and V' of the same complex dimension n are diffeomorphic if and only if they have the same total degree, the same Euler characteristic and the same Pontryagin classes.

which is, as the name suggests, still open.

[14]That is: inducing isomorphism in homotopy groups up to dimension $n - 1$ and epimorphism in n.

9 The h-cobordism theorem.

Consider two 1-connected closed oriented n-manifolds M, N diffeomorphic by f. Construct a cylinder $M \times I$ and a diffeomorphic one, obtained by gluing $M \times 1$ to $N \times 0$ by f in $M \times I \cup_f N \times I$. This is a bordism between M and N, in which inclusions of the boundary components are homotopy equivalences. The most remarkable feature of bordism is that quite often we can reverse the implication.

33. Smale's h-cobordism theorem, [12].Let W be a bordism between M and N, a closed oriented n-manifolds, such that $N \hookrightarrow W$ and $M \hookrightarrow W$ are homotopy equivalences. If $n \geq 5$, and W is 1-connected, then W is diffeomorphic to $M \times I$ with $M \hookrightarrow W \to M \times 0 = id_M$.

We call any W fullfilling the above assumptions an h-cobordism.

To pin down reader's attention, we present the Poincaré Conjecture as an immediate consequence.

34. Let Σ_n be a 1-connected homology sphere of dimension $n \geq 6$. Then it is homeomorphic to the standard sphere $\mathbb{S}^n$.

Proof. Let $\mathbb{D}^n_1$ and $\mathbb{D}^n_2$ be two small n-disks in Σ_n. Define

$$W \stackrel{\text{def.}}{=} \Sigma_n - (\mathring{\mathbb{D}}^n_1 \sqcup \mathring{\mathbb{D}}^n_2).$$

By Seifert-van Kampen theorem, W is 1-connected.

Bear in mind that $M \hookrightarrow W$ is a homotopy equivalence if and only if $\widetilde{H_*}(W, M) = 0$ by the pair sequence in homology. And so by NDR-pair[15] and excision we have

$$H_k(W, \mathbb{S}^{n-1}_i) \cong \widetilde{H}_k(W/\mathbb{S}^{n-1}_i) \cong H_k(\Sigma - \mathring{\mathbb{D}}^n_i) = 0,$$

so we get that inclusions of boundary spheres in W are homotopy quivalences.

Now we use the h-cobordism theorem and obtain in result a diffeomorphism $g : W \to \mathbb{S}^{n-1}_1 \times I$ such that $g|_{\mathbb{S}^{n-1}_1} = id_{\mathbb{S}^{n-1}}$. This extends to a diffeomorphism $G : \Sigma_n - \mathbb{D}^n_2 \to \mathbb{D}^n \cup_{id} (\mathbb{S}^{n-1}_1 \times I) \cong D^n$. By coning this diffeomorphism we obtain a homeomorphism $\Sigma_n \to \mathbb{S}^n$.

□

[15] We can write down the first isomorphism precisely because $\mathbb{S}^{n-1}$ sits in a retractible neighborhood in W.

Returning to the h-cobordism theorem, we would like to get a sketch of proof, but we will settle for an introduction to idea of a sketch instead[16].

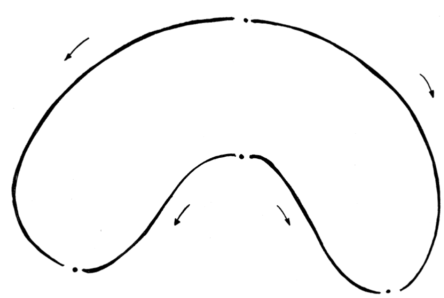
Take W, an h-cobordism between M and N and fix a Morse function $m : W \to \mathbb{R}$, distinguishing M and N with different values, say, 0 and 1. If m has no critical values, we are done by the general Morse theory: the gradient flow gives an diffeomorphism between W and $M \times I$, not moving the bottom level. Therefore a good bet would be to try to improve initial function so that it lost all critical points.

Pictures go here!

35. Cancellation lemma. Let $m : W \to \mathbb{R}$ be a Morse function with only two critical points x, y fulfilling $m(x) < m(y)$ and $ind_x(m) + 1 = k + 1 = ind_y(m)$. Furthermore let the boundaries of their upper and lower discs $\partial\mathbb{D}_{x,u}$ and $\partial\mathbb{D}_{y,l}$ intersect transversally in only one point. Then there exists a Morse function $m' : W \to \mathbb{R}$ with no critical points.

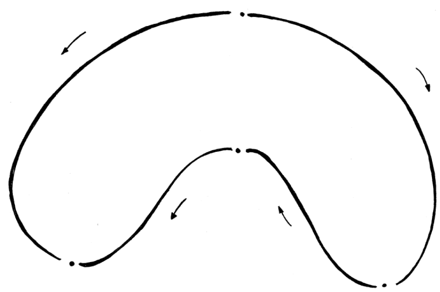
Idea is to reverse the gradient flow along the single trajectory linking two critical points and produce w new function with such an altered gradient.

Consider an example: draw a Morse function for the regular CW-decomposition of a sphere (two 0-cells, two 1-cells, two 2-cells, each glued by homeomorphism).

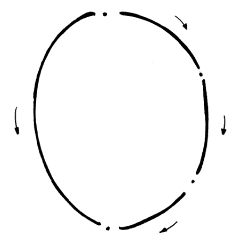


Now reversing the gradient flow we get

[16]Note that we present the function/flow approach due to Milnor and Morse, [9]. Smale's original idea developed at the beaches of Rio was to use handlebody decompositions.



and consequently,



a Morse function giving the minimal (single 0-cell and single 2-cell) CW-decomposition.

Therefore we need to rearrange the critical values so that we can cancel out all of them. The easy part is to arrange the values in accordance with their indices. The hard part is to assure transversality. Now we shall observe, where does the h-cobordism assumption come in.

Let $m : W \to \mathbb{R}$ be a Morse function with only two critical points x, y fulfilling $m(x) < m(y)$ and $ind_x(m) + 1 = k + 1 = ind_y(m)$. W is therefore homotopy equivalent[17] to $M \cup \mathbb{D}^k \cup \mathbb{D}^{k+1}$. Stop one step earlier, define W' as $W' \overset{\text{def.}}{=} M \cup \mathbb{D}^k$ and write the exact sequence of a triple (M, W', W)

$$0 = H_{k+1}(W, M) \to H_{k+1}(W, W') \overset{\cong}{\to} H_k(W', M) \to H_k(W, M) = 0.$$

More precisely it's the only part where anything non-zero happens. The middle arrow is - algebraically - an isomorphism of two copies of $\mathbb{Z}$. Topologically, take class of $k+1$-disk $[\alpha]$, a generator in $H_{k+1}(W, W')$ and count intersection

[17] We left the manifold category to avoid some cumbersome notation irrelevant to the algebra in working; but bear in mind that everything takes place in ambient manifolds that can be deformation retracted to our spaces.

number of it's boundary $\mathbb{S}_1^k$ with $\mathbb{S}_1^{n-k-1}$, the "free" part of the handle generating $H_k(W', M)$[18]. This gives a cohomology class in $H^k(W', M)$, dual[19] to the image of $[\alpha]$ in $H_k(W', M)$ by considered isomorphism, hence the intersection number is ± 1. This is only part of success, since we need to know that the spheres intersect in exactly one point, not that counting signs of intersections gives one. But it is a good start because of

36. Whitney's trick. In the above setting, we can cancel out each pair of intersection points with oposite signs, provided that h-cobordism is of dimension ≥ 5.

We proceed as follows. Take two points of intersection of opposite signs. They can be linked by a smooth curve in $\mathbb{S}_1^k$ and with another curve in $\mathbb{S}_2^{k-1}$. By 1-connectednes, resulting circle bounds a disk in W. By the dimension assumption, this disc can be embedded in W. Now cut $\mathbb{S}_1^k$ along (some neighborhood of) the curve and push the resulting boundary of the curve in $\mathbb{S}_2^{k-1}$. Now we can glue another disk along the boundary, obtaining 1) a sphere which is 2) homologous to $\mathbb{S}_1^k$ and 3) without the points of intersection we started with and 4) without any new intersections. Iterating this procedure and using Cancellation lemma establishes the h-cobordism theorem.

It is amazing to see that the only dimension-sensitive part of the proof is the embedding of 2-disk. But the assumption is crucial - h-cobordism theorem fail differentially (but not topologically) in dimension 4. Observe that one can choose an embedded circle, but cannot avoid the disk we glue in to have (discrete) self-intersections - an avatar of restrictions in Whitney's embedding theorems. A brilliant idea of Freedman is to "push" the intersections to infinity, to the boundary of a disk, obtaining an embedding, but at a price: this yields continous, not smooth embedding.

As for the dimension 3, there is again an embedded circle, but the killing disk will have 1-dimensional intersections, which cannot be pushed away. We can still prove something, a famous Dehn's Lemma: for a 3-manifold with boundary, W and an embedded loop γ, homotopically nontrivial in ∂W, but trivial in whole W, there exists another loop μ that is nontrivial in ∂W and bounds an embedded disk in W. The loop μ is homotopically unrelated to γ, so the lemma is of no use from surgical point of view. However, the reader

[18] Again, both spheres live in submanifold $m^{-1}(v)$, for some v, therefore it makes sense to count their intersections.

[19] Not Poincaré dual, just $H^k(W', M) \cong \mathrm{Hom}(H_k(W', M), \mathbb{Z})$-dual.

is referred to [6] to witness how this lemma underlies all attempts to classify 3-manifolds.

Also, note that the h-cobordism theorem can be harmlessly generalized to manifolds with boundary and bordism rel boundary but fails if we don't assume 1-connectednes. In that case we have

37. S-COBORDISM THEOREM.[20] Let W be a bordism between M and N, closed oriented n-manifolds, such that $N \hookrightarrow W$ and $M \hookrightarrow W$ are simple homotopy equivalences. If $n \geq 4$ then W is diffeomorphic to $M \times I$ with $M \hookrightarrow W \to M \times 0 = id_M$.

where two CW-complexes X and Y are said to be simply homotopy equivalent if X can be collapsed (retracting cells with boundary not meeting any other cell in some point one after another) to some subcomplex, which in turn can be (cellularly) embedded onto some subcomplex, to which Y collapses.

10 How to use the h-cobordism theorem to classify manifolds.

As we saw, in many cases in high dimension "being h-cobordant" is equivalent to "being diffeomorphic". So we want to check, whether M and N, two closed oriented 1-connected $n-1$-manifolds are h-cobordant.

One obvious condition is that M and N must be homotopy equivalent, since inclusions into h-cobordism will provide an equivalence. But since we want to show a diffeomorfism between them, they better be homotopy equivalent already.

Another obvious condition is that they must be bordant. This is detected by characteristic classes.

38. Two manifolds are bordant if and only if all Pontriyagin and Stiefel-Whitney numbers agree.[21]

Observe that the characteristic numbers are not, in general, invariants of homotopy type, despite the fact that characteristic classes may be defined in purely homotopical terms. We will elaborate on that later.

[20] Reference needed.

[21] Reference needed. Also, in Milnor-Stasheff "Characteristic Classes", on page 53 there's a theorem of Thom stating that only equalities of Stiefel-Whitney numbers is needed.

Assume from now on, that M and N fit into bordism W. To make it an h-cobordism, we need to improve it twofold:

39. Kill fundamental group of W.

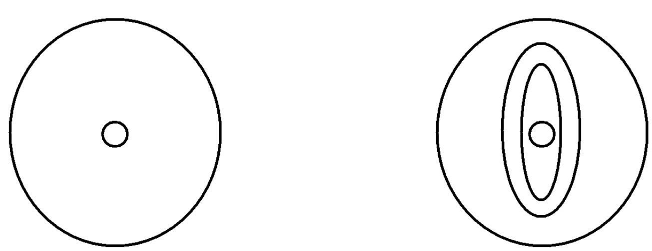
Start with an easier setting and consider a connected finite CW-complex X. We want to kill $\pi_1(X)$ by adding two-cells. Regard a map $\gamma : S^1 \to X$ representing a generator of $\pi_1(X)$ and glue in a disk, obtaining $X' = X \cup_\gamma \mathbb{D}^2$ which has fundamental group isomorphic to $\pi_1(X)/\langle[\gamma]\rangle$, division by the normal subgroup generated by $[\gamma]$. Since the fundamental group is finitely generated, eventually we obtain a 1-connected CW-complex. We now want to use this procedure, but without leaving the smooth category.

Consider W, $\dim W > 3$ and a generator in $\gamma : \mathbb{S}^1 \to W$. By the Whitney embedding theorem we can approximate γ by an embedding $\gamma : \mathbb{S}^1 \hookrightarrow W$. Now consider the normal bundle. Since W is orientable, the normal bundle is trivial. Applying the tubular neighborhood theorem gives an embedding $\widetilde{\gamma} : \mathbb{S}^1 \times \mathbb{D}^{n-1} \hookrightarrow W$.

Now take a cylinder over W and glue a cell on the top, using $\widetilde{\gamma}$. This results in $Z \overset{\text{def.}}{=} W \times I \cup_{\widetilde{\gamma}} (D^2 \times D^{n-1})$[22].
The boundary of this new manifold is $\partial Z = W \times \{0\} \cup M \times I \cup N \times I \cup (W \times \{1\} - f(\mathbb{S}^1 \times \mathbb{D}^{n-1})) \cup \mathbb{D}^2 \times \mathbb{S}^{n-2}$. Take $T \overset{\text{def.}}{=} (W \times \{1\} - f(\mathbb{S}^1 \times \mathbb{D}^{n-1})) \cup \mathbb{D}^2 \times \mathbb{S}^{n-2}$ and observe that this is a bordism on it's own, with boundary $\partial T = M \bigsqcup N$. There is a homotopy equivalence $T \simeq W \cup \mathbb{D}^2$. Using the same argument as above for the finite CW-complex we get $\pi_1(T) = \pi_1(W)/\langle[\gamma]\rangle$ and - eventualy - a 1-connected bordism.

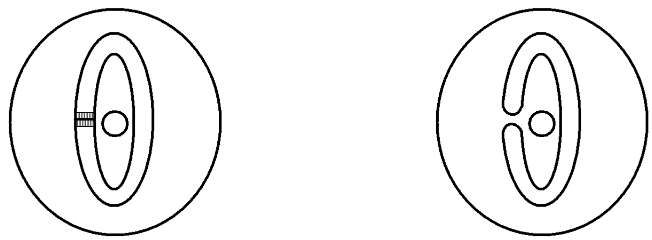
We say that Z is obtained by attaching a handle to $W \times I$, and that T is obtained from W by performing surgery on loop γ.

An example seems handy. Since we are limited in what we can draw, we will show an example of surgery along with it's limitations. In dimension 2, consider a genus one torus.



[22]This is a manifold "with corners". It is smoothable, but we don't need it here.

We only draw a planar section, since it will be sufficient. A surgeon proceeds as follows. He crosses the torus with unit interval and chooses a spherical homology generator he wants to kill, together with it's tubular neighborhood.



After cut-and-paste part, we are left with a bordism between a torus and a sphere.

Observe that in this case performing surgery on one generator of fundamental group kills the other generator as well. As this is a fenomen of topological nature, occurring only in dimension 2, it is easily translated to algebra and may be related to some problems which we will encounter when dealing with surgeries in middle dimension later on.

Having the fundamental group dispatched, we turn to the second part of the h-cobordism definition.

40. $H_*(W, M) = H_*(W, N) = 0$

From the homology long exact sequence of the pair (W, M) we get a map $H_2(W) \to H_2(W, M)$ which is onto, since M is 1-connected. And because W is now also 1-connected, by Hurewicz Theorem $H_2(W) \cong \pi_2(W)$. That means for every element $\alpha \in H_2(W, M)$ we can find a map $f : S^2 \to W$ - by Whitney Embedding Theorem and dimension assumptions even an embedding - such that $f_*[S^2] = \alpha$. A homology class represented this way is often called 'spherical'. As we want to repeat the surgery operations performed on loops, the question arises whether this embedding has trivial normal bundle.

We formulate a general remark. Since every vector bundle is trivial over contractible subset, for constructing a rank k oriented vector bundle over n-sphere it is sufficient to glue two rank k trivial bundles over n-disk along equator, which in turn defines a map from equator to $SO(k)$. Since two homotopic maps will yield isomorphic bundles, we get

41. Classification of vector bundles over spheres.

$$\mathrm{Vect}_k(\mathbb{S}^n) \cong \pi_{n-1} SO(k).$$

We used already that $\mathrm{Vect}_k(\mathbb{S}^1) \cong \pi_0 SO(k) = 0$, exactly nonorientability being the only obstruction to triviality.

In two dimensional case we have $\mathrm{Vect}_k(\mathbb{S}^2) \cong \pi_1 SO(k)$ so for $k \geq 3$ we get $\pi_1 SO(k) = \pi_1(\mathbb{R}P^3) = \mathbb{Z}_2$.

In our setting this obstruction vanishes if W carries a spin structure[23], but we need to develop a general framework.

[24] Before we do this, as a concrete illustration on how rigid the manifold setting is when one comes to a problem of killing homology classes, consider the Kummer surface V_4. After some effort is made, one can show that it's intersection form is

$$E_8 \oplus E_8 \oplus 3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and we recognize the intersection form of a torus $\mathbb{S}^2 \times \mathbb{S}^2$ in the last summand. Regarding the procedure above, we could hope to find three embedded tori representing the last part of the intersection form and kill them, obtaining a manifold V_c, the Kummer surface crippled, with intersection form $E_8 \oplus E_8$. Unfortunately, by the Donaldson work on 4-manifolds, we have a following theorem

42. No smooth manifold realizes any nonzero even positive definite intersection form.

And so we see that performing surgery in this case is somehow prohibited.[25]

11 Controlled spaces.

We have to consider a smaller class of bordisms. Before we start, a dictionary.

Recall that a monoid is a set A with associative action and a neutral element. Common examples are

[23] I believe this was stated as "if and only if", but since we work with one sphere at a time, we don't use that the spin is given globally; also, spin structures were put off for a few days from this point on.

[24] I thought that the reader might have developed some feelings towards V_4 at this point, so I'd put it again as an example.

[25] I have no time to put all the details about this example, I just remember reading it somewhere, so I won't be able to discuss it when it comes to the Θ-classes; maybe You will want to include it

- natural numbers
- isomorphism classes of R-modules with direct sum
- isomorphism classes of vector bundles over X with Whitney sum
- diffeomorphism classes of connected oriented smooth manifolds with connected sum.

Label certain objects as trivial, and we say that

43. [26]A and B are *reduced stably isomorphic* if and only if there exist T_1, T_2, trivial objects, such that $A + T_1 = B + T_2$. We reserve the name *stably isomorphic* for the case $T_1 \cong T_2$.

We list examples of stably isomorphic classes

- $\mathbb{Z}_p$, natural numbers mod multiplicity of p
- $\widetilde{K}_0 R$, group of isomorphism classes of projective R-modules mod free modules
- $\widetilde{K}_0(X)$, vector bundles over X mod trivial bundles

The last notion will be of use for us.

44. Any manifold M is naturally equiped with a stable normal bundle, $\nu(M)$, a well defined stable isomorphism class of normal bundles arising from embeddings into high $\mathbb{R}^n$.

Let X be a CW-complex and let $E \to X$ be a stable vector bundle.

45. For a closed manifold M, a map $f : M \to X$ is called a *normal map*, or a (X, E)*-structure*, if f^*E is stably isomorphic to $\nu(M)$.

46. An (X, E)-structure on M is called called a *normal* k*-smoothing* if $f : M \to X$ is a $(k+1)$-equivalence, ie.

- $f_* : \pi_i(M) \to \pi_i(X)$ is an isomorphism for $i \leq k$,

[26] I put here the notation stable/reduced stable as we generally use; I believe there was some confusion during the lecture, but if prof. Kreck changed the names deliberately, feel free to change this.

- $f_* : \pi_{k+1}(M) \to \pi_{k+1}(X)$ is an epimorphism.

For $k = \infty$, ie. when f is a homotopy equivalence, it is simply called a smoothing.

Assume now that M, M' and W are endowed with (X, E)-structures such that the following diagram commutes

$$\begin{array}{ccccc} M & \longrightarrow & W & \longleftarrow & M' \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle F} & \swarrow{\scriptstyle f'} & \\ & & X & & \end{array}$$

and moreover that f and f' are smoothings. Then it is clear that W is an h-cobordism if and only if F is a smoothing as well. Hence our problem is reduced to the following: can we modify W by a sequence of surgeries so that $F : W \to X$ is a smoothing?

Using the mapping cylinder of F we may assume that F is an inclusion. Suppose we already have $\pi_i(X, W) = 0$ for $i \leq r \leq \frac{\dim W}{2}$. From the long exact sequence of a pair and relative Hurewicz theorem we have

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & H_{r+1}(X, W) & \longrightarrow & H_r(W) & \xrightarrow{F_*} & H_r(X) & \longrightarrow & H_r(X, W) = 0 \\ & & \cong\uparrow & & & & & & \\ & & \pi_{r+1}(X, W). & & & & & & \end{array}$$

In particular, F_* is an epimorphism and each class $\alpha \in \ker F_*$ is represented by a sphere $\mathbb{S}^r = \partial \mathbb{D}^{r+1}$ such that there exist a commutative diagram

$$\begin{array}{ccc} \mathbb{D}^{r+1} & \longrightarrow & X \\ \subset\uparrow & & \uparrow{\scriptstyle F} \\ \mathbb{S}^r & \longrightarrow & W. \end{array}$$

In particular, $F^*E|_{\mathbb{S}^r}$ is trivial. Thus, for $n \gg 1$ we have $\nu(W \subset \mathbb{R}^n)|_{\mathbb{S}^r}$ trivial. But

$$\nu(\mathbb{S}^r \subset W) \oplus \underbrace{\nu(W \subset \mathbb{R}^n)}_{trivial} = \underbrace{\nu(\mathbb{S}^r \subset \mathbb{R}^n)}_{trivial},$$

and so $\nu(\mathbb{S}^r \subset W)$ is stably trivial. By using the following

47. If $E \to S^r$ is a vector bundle of dim $s > r$ then a stable trivialisation is, up to isotopy, the same as a trivialisation.

we may assume that $\nu(\mathbb{S}^r \subset W)$ is trivial. If we moreover assume that $2r < \dim W$ then the sphere $\mathbb{S}^r$ can be chosen to be smoothly embedded, and hence we may perform the surgery.

Repeating the same argument for each element of $\ker F_*$ we may replace (W, F) with (W', F') such that F'_* is an isomorphism. Then we have an exact sequence

$$\begin{array}{ccccccc} H_{r+1}(W') & \longrightarrow & H_{r+1}(X) & \longrightarrow & H_{r+1}(X, W') & \longrightarrow & 0 \\ & \nwarrow & \nearrow_{\cong} & & & & \\ & H_{r+1}(M) & & & & & \end{array}$$

and we may conclude that $H_{r+1}(X, W') = 0$.

The considerations above can be summarized with the following theorem.

48. Let F be an (X, E)-structure on the bordism W between 1-connected manifolds M and M', extending normal r-smoothings f and f' over M and M' respectively, for $r > \frac{\dim M - 1}{2}$. Then, by a sequence of surgeries, we may replace (W, F) with a new bordism W' and (X, E)-structure F' on W' such that

- F' extends f and f',
- F' induces isomorphism on π_i for $i < \frac{\dim W' - 1}{2}$.

12 Stable classification.

Recall the discusion about stability. From now on, the most important to us will be the manifold setting.

49. M and N, two closed connected $2q$-manifolds are *stably diffeomorphic* (homeomorphic, homotopy equivalent) if and only if there exist r, s such that $M\#_r(\mathbb{S}^q \times \mathbb{S}^q) \cong N\#_s(\mathbb{S}^q \times \mathbb{S}^q)$.

They are called *reduced stably isomorphic* if $r = s$. Classes of this relation will be called *stable classes*.

50. In dimension two notions of stability and reduced stability are the same.

51. Two stably diffeomorphic manifolds are reduced stably diffeomorphic if and only if their Euler classes agree.

We will aim for the following.

52. Stable classification theorem.[27] Let (M^{2q}, f) and (M'^{2q}, f') be normal $(q-1)$-smoothings in (X, E). If (M^{2q}, f) is normally bordant to (M'^{2q}, f') then M and M' are stably diffeomorphic.

Before we engage the proof, let's consider an application.

53. For M, M' two smooth, 1-connected 4-manifolds, with odd intersection forms S_M and $S_{M'}$. M and M' are stably diffeomorphic if and only if their signatures agree.

Proof. First of all we need to choose some (E, X). Take $ESO \to BSO$, the classifying space for oriented vector bundles[28]. For a smooth oriented manifold the classifying map of the stable normal bundle is called normal Gauss map $\nu : M \to BSO$.

Because the intersection forms are odd, the Gauss maps are 2-equivalences (i.e. a 1-smoothings). Now observe that (M, ν) and (M', ν') are normally bordant if and only if they are bordant. First of all, we need only to extend one of the Gauss maps over the bordism

54. Dold-Thom.[29] An oriented stable vector bundle $E \to X$, over a 4-CW complex is determined by $p_1(E)$ and $w_2(E)$.

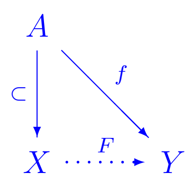
We proceed as follows. Suppose we have F, an extension of f over W and restrict F to M'. By definition of the normal bundle and since the collar of the boundary is a trivial bundle, we have $p_1 F^* ESO|_{M'} = -p_1(TW|_{M'}) = -p_1(M') = -\operatorname{sign} M' = p_1\nu(M')$, and similarly for the Stiefel-Whitney class. Thus extending one Gauss map gives on the other end of the bordism a bundle necessarily isomorphic to the original one.

[27] Reference needed.

[28] The space BSO has the property that for each CW complex Z the isomorphism classes of stable oriented vector bundles over Z are in one-to-one correspondence to the homotopy classes of maps $[Z, BSO]$, exactly by pulling back the bundle ESO.

[29] Reference needed.

And how does one extend a map? As before, we use the obstruction theory. For a inclusion and a map f



we ask for an extension, F.

55. There is a sequence of cohomology classes $\Theta_i \in H^i(W, M; \pi_{i-1}(BSO))$ such that $\Theta_i = 0$ if and only if the extension F exists, compare [16].

As the homotopy groups of BSO are well known, we have that

- Θ_1 lives in a trivial group (having trivial coefficients).
- Θ_3 lives in a group dual to the $H_1(W, M; \pi_3(BSO))$ and this can be surgered to 0.
- Θ_4 lives in a group dual to $H_0(W, M; \pi_4(BSO))$ and this is trivial, since both spaces are connected[30] .

So to use our theorem, we only need to check that M and M' are bordant, but this follows from the Rokhlin's theorem about signatures. □

For non-1-connected case, one can get the following:

56. Let M and M' be closed oriented 4-manifolds with $\pi_1(M) \cong \pi_1(M')$ and such that there exist maps $S^2 \to M$ and $S^2 \to M'$ with odd self intersection number. Then M and M' are stably diffeomorphic if and only if:

- $\operatorname{sign} M = \operatorname{sign} M'$,
- the classifying maps of the universal coverings $u : M \to K(\pi_1, 1)$ and $u' : M' \to K(\pi_1, 1)$ (inducing isomorphisms on the fundamental groups) fulfill $u_*[M] = u'_*[M']$ in $H_4(K(\pi_1, 1))$.

[30] There is a slight problem here, we don't know why the second obstruction wanishes.

Moreover, one gets a stable diffeomorphism

$$M\#_r(S^q\times S^q)\cong M'\#_s(S^q\times S^q)$$

such that $f_*:\pi_1(M)\to\pi_1(M')$ is $u'_*\circ(u_*)^{-1}$.

Now we want to prove the stable diffeomorphism theorem.

Proof. Let (M^{2q},f) and (M'^{2q},f') be normally bordant by (W^{2q+1},F). By the previous considerations, we can assume that $H_i(W,M)\cong H_i(W,M')=0$ if $i<q$. In the diagram

$$\begin{array}{ccccccc}
H_q(W) & \xrightarrow{e} & H_q(W,M) & \xrightarrow{d} & H_{q-1}(M) & \longrightarrow & H_{q-1}(W)\\
\uparrow g & & \uparrow\cong & & \uparrow c & & \\
\pi_q(W) & \xrightarrow{f} & \pi_q(W,M) & \xrightarrow{b} & \pi_{q-1}(M) & \xrightarrow{a} & \pi_{q-1}(W)\\
 & & & & \cong\searrow & & \swarrow\\
 & & & & & \pi_{q-1}(X) &
\end{array}$$

assumed bottommost isomorphism implies that a is injective, so b is zero. Then cb is zero, and so must be d, by the Hurewicz isomorphism between relative groups. Now e must be onto, and because f is onto, eg must be onto as well[31]. So we get that each class in $H_q(W,M)$ is spherical, sphere coming from the interior of W.

Pictures go here!

So we proceed as before: represent a class x and choose an embedding $S^q\times D^{q+1}\hookrightarrow\mathring{W}$.

We want to kill x exactly by making it a boundary class. This will change the boundary, but remember that we are only interrested in stable class of M. Let A be our embedded full torus together with a bridge $D^{2q}\times I$ to the boundary manifold M. Now substract the torus to get $W_\#\overset{\text{def.}}{=}W-\mathring{A}$ with boundary $\partial W_\#=M'\sqcup M\#S^q\times S^q$. Relabel the deformed boudary component by $M_\#$. In homology we get

$$H_q(W^\#,M^\#)\cong H_q(W,M)/_{\langle x\rangle}.$$

But by killing this element x we may have introduced a new nontrivial class $y\in H_q(W_\#,M')\cong H^{q+1}(W_\#,M_\#)$ represented by $*\times\mathbb{S}^q\subset\mathbb{S}^q\times\mathbb{S}^q\subset$

[31] Ufff...

$\mathbb{S}^q \times \mathbb{D}^{q+1}$, i.e. $H_q(W_\#, M') \cong H_q(W, M') + \mathbb{Z} \cdot y$, which frustrates our knavish tricks.

But observe that $H_q(W_\#, M') \cong H^{q+1}(W_\#, M_\#)$ and the latter is free. So take a basis $x_1, \ldots, x_r$ of $H_q(W_\#, M')$ and proceed as before: each x_i is spherical, so represent it by an embedded sphere, connect it with the M' component of the boundary, and substract interiors of the resulting sets. We obtain a (normal) bordism $\overline{W}$ between $M_\#$ and (new) $M'_\#$ with $H_q(\overline{W}, M'_\#)$ killed. We need to check what harm did we do to $H_q(\overline{W}, M_\#)$.

Notice that

$$H_{q+1}(\overline{W}, M' \#_r \mathbb{S}^q \times \mathbb{S}^q) \cong H_{q+1}(W_\#, M') \oplus \ker(([\mathbb{S}^q]_1, \ldots, [\mathbb{S}^q]_r) \mapsto (x_1, \ldots, x_r))$$

the last summand equal 0 because the generators are free. By Poincaré Duality and Universal Coefficients Theorem

$$0 = H_{q+1}(\overline{W}, M' \#_r \mathbb{S}^q \times \mathbb{S}^q) \cong H^q(\overline{W}, M_\#) \cong H_q(\overline{W}, M_\#)$$

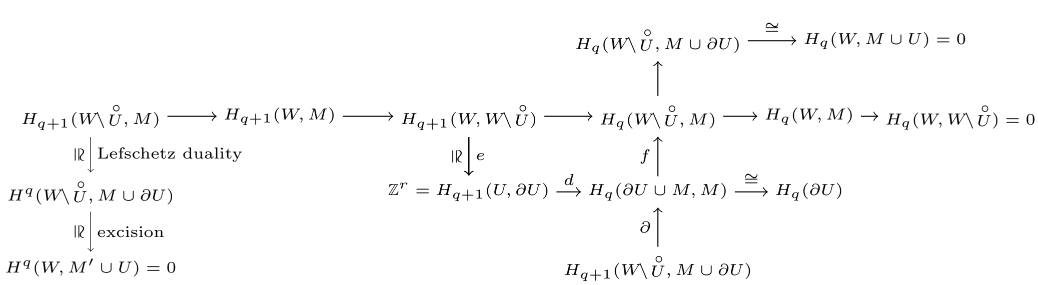
Thus the pair $(\overline{W}, M_\#)$ remains q-connected and the proof is finished. □

13 Obstructions in odd-dimensional case and l-monoids.

Again regard normal maps for two manifolds (M, f, α) and (M', f', α') and a bordism (W^{2q+1}, F, β) between them which is simply connected and its homology rel M and M' vanishes

$$H_i(W, M) \cong H_i(W, M') = 0, i < q.$$

Choose x_j represented by disjoined embeddings of $\mathbb{S}^q \times \mathbb{D}^{q+1}$ such that they generate both $H_q(W, M)$ and $H_q(W, M')$. Set $U = \bigsqcup(\mathbb{S}^q \times \mathbb{D}^{q+1})$. Consider following enormous diagram:



The horizontal sequence is the long exact sequence of the triple $(W, W\backslash U, M)$. The same goes for the vertical sequence (for triple $(W\backslash U, M \cup \partial U, M)$).

We know that $H_q(U)$ is isomorphic to $\mathbb{Z}^r \oplus \mathbb{Z}^r$, the former summand generated by $(S^q \times \{*\})_j$, the latter by $(\{*\} \times S^q)_j$. We denote q-th homology group of U by $H_{(-1)^q}(\mathbb{Z}^r)$. The intersection form of U then can be represented by the following matrix

$$\begin{pmatrix} 0 & \mathrm{Id}_r \\ (-1)^q \,\mathrm{Id}_r & 0 \end{pmatrix},$$

57. Let V denote image of the homomorphism $\operatorname{Im} \partial \subset H_q(U) = H_\epsilon(\mathbb{Z}^r)$. Then V is a direct summand of rank r.

Consider pairs $(H_\epsilon(\mathbb{Z}^r), V)$, where V is a half rank direct summand and introduce relation

$$(H_\epsilon(\mathbb{Z}^r), V) \sim (H_\epsilon(\mathbb{Z}^{r+s}), V \oplus \mathbb{Z}^s \times \{0\})).$$

Define

$$R_\epsilon(U) \overset{\text{def.}}{=} \langle \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \rangle.$$

This first, "flipping" matrix corresponds to carrying out a surgery. Then $R_\epsilon(U)$ is group of isometries of $H_\epsilon(\mathbb{Z}^r)$ and the relation becomes

$$(H_\epsilon(\mathbb{Z}^r), V) \sim (H_\epsilon(\mathbb{Z}^r), \varrho(V))$$

for some $\varrho \in R_\epsilon(U)$.

Now define ℓ-monoid ℓ_{2q+1} to be monoid of the equivalence classes of $(H_\epsilon(\mathbb{Z}^r), V)$ together with the orthogonal sum. Wall groups

$$L_{2q+1} \overset{\text{def.}}{=} \{[H^r_\epsilon, V] \colon H^r_\epsilon \text{ vanishes on } V\}$$

occur as a submonoids of ℓ-monoids.

58. Suppose we have normal bordism (W, F, β). Then element

$$\Theta(W, F, \beta) \overset{\text{def.}}{=} [H_\epsilon(\mathbb{Z}^r), V] \in l_{2q+1}$$

is well defined and if (W, F, β) is bordant relative boundary to (W', F', β') then $\Theta[W, F, \beta] = \Theta[W', F', \beta']$

59. An element in $[H_\epsilon(\mathbb{Z}^r), V]$ is called *elementary* if and only if $(H_\epsilon(\mathbb{Z}^r), V)$ is equivalent to $[H_\epsilon(\mathbb{Z}^s), V']$ such that $\mathbb{Z}^s \times \{0\} + V' = \mathbb{Z}^s \oplus \mathbb{Z}^s$.

Note that if $[H_\epsilon(\mathbb{Z}^r), V] \in L_{2q+1}$ then an object is elementary if and only if it is 0.

60. Regard (W, F, β) as before. Then (W, F, β) is bordant rel boundary to an h-cobordism if and only if $\Theta(W, F, \beta)$ is elementary.

Proof. We aim to show that the middle arrow in horizontal sequence is an isomorphism, for that implies vanishing of $H_{q+1}(W, M)$ and $H_q(W, M)$.

Consider $\Theta(W, F, \beta)$ equivalent to an elementary element $(H_\epsilon(\mathbb{Z}^r), V)$. One can show that the choices involved in producing the equivalent elementary element can be realised by

- adding new generators
- carrying out surgery
- changing the generators

Thus we produce (W', F', β') such that $\mathbb{Z}^s \times \{0\} \oplus V = \mathbb{Z}^s \oplus \mathbb{Z}^s$. The image $\operatorname{Im} d = \{0\} \times \mathbb{Z}^r$, so after one surgery it becomes $\mathbb{Z}^r \times \{0\}$. Observe that now composition fde is an isomorphism, and thus $H_{q+1}(W, W \backslash U) \cong H_q(W \backslash U, M)$. □

13.1 The non-1-connected case

Now we want to consider the non-1-connected case and denote $\pi_1 = \pi_1(W)$. Note that the h-cobordism theorem is wrong in this case. But if we have an h-cobordism W then there is an obstruction, called the Whitehead torsion $\tau(W) \in Wh(\mathbb{Z}[\pi_1])$. If it vanishes, the h-cobordism theorem holds. Begin with the $\mathbb{Z}[\pi_1]$-module, and take direct limit of groups

$$GL(\mathbb{Z}[\pi_1]) = \varinjlim_n Gl(\mathbb{Z}[\pi_1], n).$$

First algebraic K-theory group of the group ring $\mathbb{Z}[\pi_1]$, $K_1(\mathbb{Z}[\pi_1])$ is defined as an abelianization of this limit

$$K_1(\mathbb{Z}[\pi_1]) \stackrel{\text{def.}}{=} GL(\mathbb{Z}[\pi_1])_{abel.}$$

The Whitehead group is a quotient of $K_1(\mathbb{Z}[\pi_1])$ by subgroup generated by elementary matrices. Differently speaking $Wh(\mathbb{Z}[\pi_1]) = K_1(\mathbb{Z}[\pi_1])/_{\langle \pm g\rangle}$, division by inclusions of elements of π_1 into $GL(Z[\pi_1])$.

61. Suppose that $\dim W \geq 6$. Then a h-cobordism is a cylinder if and only if $\tau(W) = 0$.

For $\pi_1 \neq 0$ when is a bordism W an h-cobordism?
For all $i \geq 2$ we have:

- $\pi_1(M) \cong \pi_(M') \cong \pi_1(W)$
- $\pi_i(W, M) = \pi_i(W, M') = 0$
- $H_i(\widetilde{W}, \widetilde{M}) = H_i(\widetilde{W}, \widetilde{M}') = 0$

Here the tildes indicate the universal covering. Working there we are back in the simply connected case.

In our big diagram we had the union of our generators U. If we lift that to the universal covering we get $n = \operatorname{rank} \pi_1$ copies of U:

$$\widetilde{U} = \bigsqcup_{i=1}^{n} (\mathbb{S}^q \times \mathbb{D}^{q+1})_i.$$

Thus

$$H_q(\partial \widetilde{U}) = \oplus H_\epsilon(\mathbb{Z}^r) \otimes_{\mathbb{Z}} \mathbb{Z}[\pi_1].$$

Analogously we will denote this homology group by $H_\epsilon(\mathbb{Z}[\pi_1]^r)$. We know that $V \subset H_\epsilon(\mathbb{Z}[\pi_1]^r)$ is a free $\mathbb{Z}[\pi_1]$-module of rank r. Then the ℓ-monoids are defined

$$\ell_{2q+1}(R) \stackrel{\text{def.}}{=} \{(H_\epsilon(R^r), V)\}/_{R_\epsilon(U)}.$$

With these definitions we get the same theorem as in the 1-connected case.

62. [32] Suppose that (W, F, β) is not necessarily simply connected. Then (W, F, β) is bordant relative boundary to an h-cobordism if and only if $\Theta(W, F, \beta) \in \ell_{2q+1}(\mathbb{Z}[\pi_1])$ is elementary.

[32] One has to be careful when doing surgery with noncompact manifold, because homology may not be finitely generated. We may proceed in this case, because it is finitely generated over $\mathbb{Z}[\pi_1]$.

14 Normal homotopy type and normal k-types.

63. M and N are *normally homotopy equivalent* if and only if there is a homotopy equivalence $f : M \to N$ such that $f^*\nu(N) \cong \nu(M)$.

We now want to consider a weaker invariant:

64. The *normal k-type* of a smooth oriented manifold is an isomorphism class of fibrations $F \hookrightarrow B_k(M) \to BSO$, satisfying two conditions:

- there is a normal k-smoothing (a lift of ν to $B_k(M)$) $\widetilde{\nu}$ of M in $B_k(M)$, that is $\widetilde{\nu}$ is a $(k+1)$-equivalence and
- $\pi_i(F) = 0$ for $i \geq k+1$.

Note that for $k > \dim M$ the normal k-type is just the normal homotopy type.

65. The normal $0-$type M is just $\{x\} \hookrightarrow BSO \to BSO$, $\pi_1 BSO$ being trivial.

66. To determine the normal 1-type of M we need to consider three cases.

- $w_2(\widetilde{M}) \neq 0$

  Then the normal 1-type is given by the trivial fibration

  $$K(\pi_1(M), 1) \times BSO$$

  with

  $$\begin{array}{ccc} & & K(\pi_1(M),1) \times BSO \\ & \overset{(u,\nu)}{\nearrow} & \downarrow \\ M & \xrightarrow{\ \nu\ } & BSO \end{array}$$

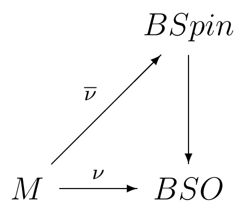
  where u classifies the universal covering of M. Now

$$\pi_2(M) \cong H_2(M) \to \pi_2(K(\pi_1(M),1) \times BSO) \cong \pi_2 BSO \cong \mathbb{Z}_2$$

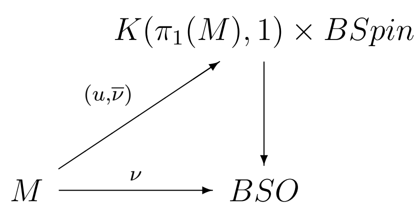
is surjective exactly when the second Stiefel-Whitney class does not vanish.

- $w_2(M) = 0$

  That is to say, when there exists a lifting



  and we can easily see that the normal 1-type is the trivial fibration



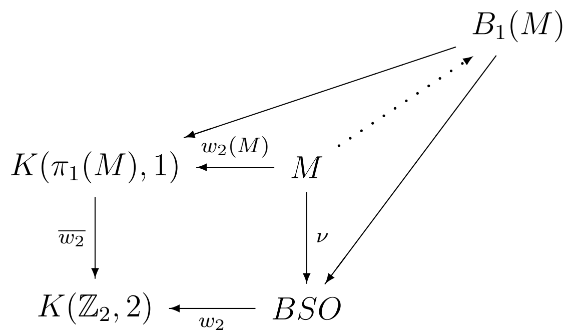
  because $\pi_2(K(\pi_1(M),1) \times BSpin) = 0$.

- $w_2(\widetilde{M}) = 0, w_2(M) \neq 0$

  We have the sequence:

$$0 \to H^2(K(\pi_1(M),1);\mathbb{Z}_2) \xrightarrow{u^*} H^2(M;\mathbb{Z}_2) \xrightarrow{p^*} H^2(\widetilde{M};\mathbb{Z}_2)$$

  and by the assumption we know that $p^* w_2(M) = 0$. So there is (unique) $\overline{w_2} \in (u^*)^{-1}(w_2(M) \subset H^2(K(\pi_1(M),1);\mathbb{Z}_2)$ and bearing in mind the relation $H^q(X,R) \cong [X, K(R,q)]$ we get a pullback diagram



with $B_1(M) \to BSO$ the normal 1-type. The conclusion is that it depends only on $\pi_1(M)$ and $w_2(M)$.

67. Given a fibration $B \to BSO$ we define the *B-bordism group*

$$\Omega_n^B \overset{\text{def.}}{=} \{(M, f) | M \text{ closed, } f : M \to B \text{ and } p \circ f = \nu\}/_{\simeq}$$

Observe that this is just reformulation of normal bordism: given (X, E), classify it by a map $X \to BSO$. Composing apriopriate maps, we get that normal bordism groups in (X, E) are Ω_n^E.

Our main result is the following.

68. Let M and N be compact manifolds, $f : \partial M \to \partial N$ a diffeomorphism and suppose M and N have the same normal k-type B_k and normal $k-$smoothings $\widetilde{\nu}_M$ and $\widetilde{\nu}_N$ with $\nu_N \circ f = \nu_M$. We can glue both structures together not leaving the group $\Omega_n^{B_k}$ and suppose $[M \cup_f N, \widetilde{\nu}_M \cup \widetilde{\nu}_N]$ is zero element there, bounding some (W, h). Then

- if $n = 2q$ and $k = q - 1$ there is an obstruction $\Theta(W, h) \in l_{2q+1}^s(\pi_1(W))$ such that $\Theta(W, h)$ is elementary if and only if (W, h) is bordant rel boundary to an s-cobordism
- if $k \geq q$ then $\Theta(W, h)$ is an element in $L_{2q+1}(\pi_1(W))$
- if $m = 2q + 1$ and $k = q - 1$ then there exists an obstruction $\Theta(W, h)$ in $l_{2q+2}^s(\pi_1(W))$ such that $\Theta(W, h)$ is elementary if and only if (W, h) is bordant rel boundary to an s-cobordism

15 Concrete problems or good things come to those who wait.

Now consider some concrete problems:

1. Bordism of diffeomorphisms: let $f : M \to M$ be an orientation-preserving on a closed nullbordant manifold; is there a closed manifold W, with $\partial W = M$ and a diffeomorphism $F : W \to W$ extending f?

2. For each n, is there a closed n-manifold with unique smooth structure?

3. Are two homeomorphic closed homogenous spaces[33] $M = G_M/H_M$ and $N = G_N/H_N$ necessarily diffeomorphic?

4. Is there a closed manifold such that the moduli space of Riemannian metrics with positive sectional curvature is not connected?

5. Given a sequence $H_0 = \mathbb{Z}$,...,$H_n = \mathbb{Z}$ of abelian groups, decide if there is a closed oriented manifold M with $H_k(M) \cong H_k$.

6. Are two 4-dimensional homology spheres classified up to homeomorphism by their fundamental group? Which groups can be realized as their fundamental groups?

69. Let's start with question (1). Regard a pair (M, f) of an $m = 2q - 1$-dimensional manifold. We have at our disposal: $[M] \in \Omega_m$ and $[M_f] \in \Omega_{m+1}$, with $M_f \stackrel{\text{def.}}{=} M \times I/((x, 1) \sim (f(x), 0))$. Assume that both of them are nullbordant, the second one by X.

There is a natural map from M_f to $\mathbb{S}^1$, extend it to X and cut it along the preimage of any regular value. We get a bordism (rel boundary) X between two copies of W. We will aim for the setting of the Stable Classification Theorem. Start with a normal map $\nu : W \to BSO$ and by a sequence of surgeries obtain a $q-1$-smoothing $\nu' : W' \to BSO$. Take two copies of trace of this surgeries and glue them to X, to obtain X', a bordism between W' and W'. The diffeomorphism - of $W' \#_r \mathbb{S}^q \times \mathbb{S}^q$ onto itself - granted by the stable classification[34] is exactely the diffeomorphism extending f.

[33] Manifolds obtained as a quotients of some Lie groups.
[34] Using the fact that $\chi(W') = \chi(W')$...

So we conclude that the pair (M, f) is nullbordant if and only if M and M_f are.

The problem is also solved in the even case[35].

Now consider problem number (2). First of all, we are troubled, if the dimension we work in admits an exotic structure sphere, say Σ: $M\#\Sigma$ is then homeomorphic to M, but there is no general procedure to check if the sum changes the diffeomorphism class. We know that (apart from dimension 4 where the problem is still open) only S^1, S^2, S^3, S^5, S^6 and S^{12} have a unique smooth structure[36].

None the less there is the result

70. For each oriented bordism class in dimension $n \neq 4$ there is a manifold unique smooth structure.

Proof. Let M be a closed manifold and $\nu : M \to BSO$ it's Gauss map. As before, modify this by a surgery to a $(q-1)$-smoothing $M' \to BSO$ and regard $M'\#(S^q \times S^q) = N$. This is the manifold we are looking for. Let N' be N carrying a different smooth structure. Since both have Gauss maps (the same map) to $BSO:\ N \to BSO \leftarrow N'$, we are on good path for the obstruction theory we developed earlier. So, are N and N' bordant? We need to check whether the Pontryagin and Stiefel-Whitney numbers agree. The latter agree since they are homotopy invariant: homotopy equivalence induces isomorphism in cohomology and - intuitively - there is only one automorphism of $\mathbb{Z}_2$. As for Pontryagin numbers (again, intuitively) there are two automorphism of $\mathbb{Z}$, and a priori homotopy equivalence may - for example - change the sign of a characteristic number. Yet Novikov[37] was able to show that

71. the rational Pontryagin classes are homeomorphism invariants.

And thus we get W, the normal bordism between N and N'. So the obstruction to W being bordant rel boundary to a s-cobordism is $\Theta(W, \nu_W)$ in $l_{2q+1}(\pi_1(W))$. We only need to prove it elementary, and this will follow from that $N = M'\#(\mathbb{S}^q \times \mathbb{S}^q)$. □

[35] Reference needed.
[36] Reference needed.
[37] Reference needed.

16 Stably diffeomorphic manifolds and l-monoids: quadratic refinements.

First of all consider an example: $M = \mathbb{S}^5 \times \mathbb{S}^5$. Middle dimension cohomology is $H^5(\mathbb{S}^5 \times \mathbb{S}^5) \cong \mathbb{Z}x \oplus \mathbb{Z}y$. The intersection form $S_M : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{Z}$ has the form

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

We can see that $S_M(x+y, x+y) = 0$. Poincare dual of element $x+y$ is represented by the diagonal embedding $\Delta : \mathbb{S}^5 \to \mathbb{S}^5 \times \mathbb{S}^5$. It's normal bundle is isomorphic to $T\mathbb{S}^5$, which is nontrivial by the following theorem.

72. The only paralellizable spheres are $\mathbb{S}^1, \mathbb{S}^3$ and $\mathbb{S}^7$

trivialization coming from the Lie group structure inherited from $\mathbb{C}, \mathbb{H}$ and $\mathbb{O}$. The main difficulty lies in proving that no other spheres are paralellizable. This was dealt with by Adams in [1]. Consult [5], Chapter 4.B. for a good account of related topics.

So vanishing of the intersection form on a given class is not sufficient for this class to be suitable for surgery, ie. represented by an embedding $\mathbb{S}^q \times \mathbb{D}^q \to M$.

From now on, let π denote the fundamental group of M and let Λ be a group ring of π, ie. $\Lambda = \mathbb{Z}[\pi]$. Fundamental group acts on higher cohomology groups, hence they can be considered as Λ-modules. In Λ there is a natural involution $\sum a_n g_n \mapsto \overline{\sum a_n g_n} := \sum a_n g_n^{-1}$

We can now quote the following theorem.

73. Wall, [15]. Let $(\overline{\nu} M^{2q} \to B)$ be a normal B-manifold, and consider $ker(\pi_q(M) \to \pi_q(B)) =: K_q$. Then there exist a quadratic refinement of an intersection form, ie. a homomorphism $\mu : K_q \to \Lambda / \langle x - \epsilon \overline{x} \rangle$, for $\epsilon = (-1)^q$, satysfying

- $S_M(\alpha, \alpha) = \mu(\alpha) + \epsilon \overline{\mu(\alpha)}$
- $\mu(\alpha + \beta) = \mu(\alpha) + \mu(\beta) + S_M(\alpha, \beta)$
- $\mu(x \cdot \alpha) = x \cdot \mu(\alpha) \cdot \overline{x}$

where intersection form acts to $\mathbb{Z} \subset \Lambda$.[38] Moreover, if $q \geq 3$, then $\mu(x) = 0$ if and only if x is homologous to an embedding $S^q \times D^q \hookrightarrow M$.

We will now pass to the algebraic setting.

74. A *hyperbolic quadratic form* on Λ^r is a triple $H_\epsilon(\Lambda^r) = (\Lambda^r \oplus \Lambda^r, \lambda, \mu)$ such that $\lambda : \Lambda^r \oplus \Lambda^r \to \Lambda$ is ϵ-hyperbolic, ie. in some basis it is given by the matrix

$$\begin{bmatrix} 0 & I_r \\ \epsilon I_r & 0 \end{bmatrix}$$

and $\mu : \Lambda^r \to \Lambda/\langle x - \epsilon\overline{x}\rangle$ is a *quadratic refinement* of x.

In certain situations λ determines μ, e.g. if $\epsilon = 1$ and the fundamental group is trivial.

75. Consider

$$T_\epsilon^{\text{z. s.}}(\pi) := \{(V, \Theta) | V \text{ fin. gen. and free over } \Lambda, \Theta = (\lambda, \mu)\}/\simeq,$$

where $\simeq$ is the relation of 0-stable isometry, ie. $(V, \Theta) \simeq (W, \Phi)$ if and only if there exists an isometry such that $(V \oplus \Lambda^r, \Theta + 0) \cong (W \oplus \Lambda^s, \Phi + 0)$.

Observe, that if we consider (V, Θ) where V is a subspace $V \subset H_\epsilon(\Lambda^r)$ then we can take the "perpendicular" pair $(V^\perp, \Theta^\perp)$.

76. There exists a monoid map

$$b : \ell_{2q+1}(\pi) \to T_\epsilon^{\text{z. s.}}(\pi) \times T_\epsilon^{\text{z. s.}}(\pi)$$

mapping $[H_\epsilon(\Lambda^r), V] \mapsto ([V, \Theta], [V^\perp, -\Theta^\perp])$.

77. Define $\ell_{2q+1}(v, w) := b^{-1}(v, w)$ with $v = [V, \Theta]$ and $w = [V^\perp, -\Theta^\perp]$. Furthermore $\ell_{2q+1}(v) := \bigcup_{w \sim_H v} l_{2q+1}(v, w)$ where

$$v \sim_H w \iff v \oplus H_\epsilon(\Lambda^r) = w \oplus H_\epsilon(\Lambda^r).$$

We get the following properties:

[38] From now on we identify $K_q \subset \pi_q(M)$ with it's image by Hurewicz homomorphism, $K_q \subset H_q(M)$.

1. $\ell_{2q+1}(0) = L_{2q+1}(\pi)$ the group of unities of $l_{2q+1}(\pi)$, e.g. $L_{2q+1}(e) = 0$ and

2. $\mathcal{E}\ell_{2q+1}(\pi) \cong T_\epsilon^{\text{z. s.}}(\pi)$, where $\mathcal{E}\ell_{2q+1}(\pi)$ is a submonoid of $l_{2q+1}(\pi)$ consisting of elementary elements and $[H_\epsilon(\Lambda^r), V] \mapsto [V, \Theta] = v$. In particular $b([H_\epsilon(\Lambda^r)]) = (v, v)$.

78. Cancellation Theorem.

- Kreck. If $\epsilon = -1$ and π is trivial then

$$\mathcal{E}\ell_1(e) \cong l_1(e).$$

- Kreck. If $\epsilon = -1$, π is trivial and $v = w + [H_\epsilon(\Lambda^r)]$ then

$$\ell_{2q+1}(v) = \{H_\epsilon(\Lambda^r)\} =: \{e_V\}.$$

- Hambleton-Kreck. If $|\pi| < \infty$ and $v = w + [H_\epsilon(\Lambda^2)]$ then

$$\ell_{2q+1}(v) = \{e_V\}.$$

- Crowley-Sixt. If π is polycyclic-by-finite, ie. there exist a sequence $1 = \pi_0 \triangleleft \pi_1 \triangleleft \ldots \triangleleft \pi_n = \pi$ where each π_{i+1}/π_i is either cyclic or finite, and $v = w \oplus [H_\epsilon(\Lambda^{h(\pi)+3})]$, where $h(\pi)$ is the Hirsch number of π, ie. the number of infinite quotients in each sequence of the above form, then

$$\ell_{2q+1}(v) = \{e_V\}.$$

In particular we have the following

79. If M^{4k} and N^{4k} are normally $(2k-1)$-bordant, $\chi(M) = \chi(N)$ and $M \cong M'\sharp\mathbb{S}^{2k} \times \mathbb{S}^{2k}$ then, for $k \geq 2$, $M \cong N$.

And this ends the discussion of the second problem.

If we consider an example $(V, \Theta) = (\mathbb{Z}, p_1, ..., p_k)$, $\epsilon = 1$ and π is trivial then $|\ell_{2q+1}(v)| = 2^{k-1}$. As a consequence

80. There exist 8-manifolds with arbitrary large stable classes.

17 Back to our problems.

We will now take up problem (3) of lecture 13.
Regard the following situation: let k, l be coprime integers and regard $i_{k,l}$, embedding of $\mathbb{S}^1$ into $SU(3)$:

$$\begin{pmatrix} z^k & 0 & 0 \\ 0 & z^l & 0 \\ 0 & 0 & z^{-k-l} \end{pmatrix}$$

Define a 7-dimensional homogenous space $M_{k,l} \stackrel{\text{def.}}{=} SU(3)/i_{k,l}(\mathbb{S}^1)$.
Before we openly engage the problem, let's do some stamp-collecting.

- Each $M_{k,l}$ is 1-connected.
- $H^2(M_{k,l}) \cong \mathbb{Z}$, $H^3(M_{k,l}) = 0$ and $H^4(M_{k,l}) \cong \mathbb{Z}/(k^2 + kl + l^2)$.
- The normal 2-type of $M_{k,l}$ is $B_0 \stackrel{\text{def.}}{=} \mathbb{C}P^\infty \times BSpin$, provided $M_{k,l}$ is spin.

It happens that

81. $\Omega_7^B = 0$

So let $W \to B$ be a bordism between two $M_{k,l}$ and $M_{k',l'}$. Then we have an obstruction $\Theta(W) \in l_8$ and the theorem:

82. $\Theta(W)$ is elementary if and only if $\operatorname{sign} W = 0$ and

$$p_1(W, \partial W) \in H^4(W, \partial W, \mathbb{Q})$$

is mapped to zero in $H^4(W; \mathbb{Q})$.

83. Kreck-Stolz. $M_{k,l}$ is diffeomorphic to $M_{k',l'}$ if and only if:

- $k^2 + kl + l^2 = k'^2 + k'l' + l'^2 =: n_0$ and
- $kl(k + l) = k'l'(k' + l') \mod (2^5 \cdot 3 \cdot 7^{\lambda(n_0)} \cdot n_0)$, where

$$\lambda(n_0) = \begin{cases} 0 & N = 0 \mod 7 \\ 1 & \text{else} \end{cases}$$

.

They are homeomorphic if and only if:

- $k^2 + kl + l^2 = k'^2 + k'l' + l'^2 =: n_0$ and
- $kl(k + l) = k'l'(k' + l') \mod (2^3 \cdot 3 \cdot n_0)$, with lambda as before.

There is another theorem stating:

84. Kreck-Stolz. If $\operatorname{rank} H^4(M_{k,l}) < 296830876$, then homeomorphism and diffeomorphism classes coincide. But $M_{-56788,5227}$ is homeomorphic to $M_{-42652,61213}$ but not diffeomorphic.

Thus we took care of the third problem. It is interesting that those quotients will help us with the fourth problem as well.

Problem of finding a Riemannian manifold with prescribed properties is a very interesting question in differential geometry. For example - which manifolds admit a metric with positive scalar curvature? An example already considered by Hopf is $\mathbb{S}^2 \times \mathbb{S}^2$ and the question in this case is still open!

Most of the known examples are of the form $\mathbb{S}^n, n \geq 2$, $\mathbb{C}P^n, n \geq 1$ and $\mathbb{H}P^n$. In fact, those are only examples known above dimension 24. But Allof and Wallach[39] were able to show that

85. Each $M_{k,l}$ admits a metric with positive sectional curvature.

Another interesting question in differential geometry is: what is the shape of the space of metrics with prescribed properties? For example - is there a closed manifold M such that the moduli space of all metrics of positive scalar curvature up to diffeomorphism of M is not connected?

86. Kreck-Stolz. One example for such a manifold is $M_{-4638661,582656}$. Observe that $\operatorname{rank} H^4(M_{-4638661,582656}) = 411358875444559$.

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