The stable classification of 4-manifolds
Krakow August 2010
Exercises

1 Monday

1. Determine the intersection forms of the following manifolds
   (a) $\mathbb{C}P^2$,
   (b) $S^2 \times S^2$,
   (c) $M_1 \times M_2$, where M_i are manifolds of dim k and $4-k$.
2. Determine the signatures of the manifolds in the previous exercises.
3. Why is $V_k := \{(x_1, \dots, x_4) \mid \Sigma x_i^k = 0\} \subset \mathbb{C}P^3$ a manifold?
4. What is the degree of the map $H_4(V_k) \to H_4(\mathbb{C}P^3)$?
5. Find homotopy invariants of closed oriented 3-manifolds with a given fundamental group.

   Hint: follow the method of the classification of simply connected 4-manifolds presented in Lecture 1, using $K = K(\pi, 1)$ where $\pi = \pi_1(M^3)$.

   C.f. Charles Thomas Bull. A. M. S..
6. Let $b : A \times A \to \mathbb{Z}$ be a non-singular symmetric bilinear form such that there is a summand $C \subset A$ with $\mathrm{rank}(C) = \mathrm{rank}(A)/2$ and $b_{C \times C} = 0$. Show that $\mathrm{sign}(b) = 0$.

   (Call such a C a Lagrangian).
7. Let W^{4k+1} be a compact oriented $(4k+1)$-manifold with boundary ∂W, show that $H^{2k}(\partial W)$ contains a Lagrangian for the symmetric bilinear form defined by the cup-product pairing.

   What is $\mathrm{sign}(\partial W)$?

2 Tuesday

1. Compute the signature of $\mathbb{C}P^n \times \mathbb{C}P^m$.

2. Show that the stable complex tangent bundle of $\mathbb{C}P^n$ is $[(n+1)\bar{L}]$, where $\bar{L}$ is the conjugate of tautological bundle over $\mathbb{C}P^n$ and $(n+1)\bar{L}$ denotes the $(n+1)$-fold Whitney sum of $\bar{L}$ with itself.

   Hint: Compare with the case of $\mathbb{R}P^n$ and use $L^\perp$ the complement of $L \subset \mathbb{C}P^n \times \mathbb{C}^{n+1}$.

3. Compute the Chern classes and the Pontryagin classes of $\mathbb{C}P^n$.

4. Show that for n and m even, $\mathbb{C}P^n \times \mathbb{C}P^m$ and $\mathbb{C}P^{n+m}$ define linearly independent elements in the oriented bordism group.

5. Determine the L-polynomial $L_2(p_1, p_2)$ in dimension 8: this is a polynomial such that for a closed smooth oriented M^8, then

$$\text{sign}(M^8) = \langle L(p_1(TM), p_2(TM)), [M] \rangle.$$

   Recall that $L_1(p_1) = \frac{1}{3}p_1$.

6. Let Σ^n be a homotopy sphere: use the Seifert-Van Kampen theorem and the Mayer-Vietoris theorem to show that $\Sigma - (\text{int}(D^n) \cup \text{int}(D^n))$ is an h-cobordism $(W^n; S^{n-1}, S^{n-1})$.

7. Assuming the h-cobordism theorem, prove the following:

   (a) For $n \geq 6$, let W^n be a compact contractible manifold with simply connected boundary, then there is a diffeomorphism $W^n \cong D^n$.

   (b) For $n \geq 6$, if there is a homotopy sphere Σ^n which is not diffeomorphic to S^n then there is a diffeomorphism $f : S^{n-1} \cong S^{n-1}$ which does not extend to a diffeomorphism $F : D^n \cong D^n$.

8. Give an example of an 4-dimensional bordism $(W^4; S^3, S^3)$ which is not bordant rel. boundary to an h-cobordism.

3 Wednesday : Bundles over spheres

Let $SO_n \subset O_n$ be the inclusion of the special orthogonal group into the orthogonal group: there is a fibre bundle

$$SO_n \to SO_{n+1} \to S^n$$

and there is a long exact homotopy sequence associated to this fibre bundle

$$\cdots \to \pi_{i+1}(S^n) \to \pi_i(SO_n) \to \pi_i(SO_{n+1}) \to \pi_i(S^n) \to \pi_{i-1}(SO_n) \to \dots \quad .$$

Recall that $\text{Vect}_{\mathbb{R}^n}(S^{i+1}) \equiv [S^i, O_n]$, via $E \to S^{i+1}$ maps to the "clutching function" of E.

1. Show that if $i < n-1$, then $\pi_i(SO_n) \cong \pi_i(SO(n+1))$: denote the stable group $\pi_i(SO)$.

2. Determine $\pi_0(O)$, $\pi_1(SO)$ and $\pi_2(SO)$ and as many other higher groups as you can.

3. Let $E \to S^r$ be a vector bundle of rank $k > r+1$. Show that a trivialisation of $E \oplus \underline{\mathbb{R}^n} \to S^r$ defines a trivialisation of $E \to S^r$ (here $\underline{\mathbb{R}}^n$ denotes the trivial rank n bundle and $\oplus$ Whitney sum).

   As an extension, show that the trivialisation of $E \to S^r$ above is well-defined up to homotopy of vector bundle isomorphisms.

4. Let E be an oriented rank $2k$ vector bundle over S^{2k}; show that E is trivial if and only if $E \oplus \underline{\mathbb{R}}$ is trivial and the Euler class of E vanishes.

   Hint: you may assume that $\partial : \pi_{2k}(S^{2k}) \to \pi_{2k-1}(SO_{2k})$ maps a generator to the homotopy class of the clutching function for TS^{2k}.

4 Thursday: Surgery

1. Let (X, E) be a vector bundle with $\mathrm{rank}(E) >> \dim(X) \geq i+1$, let $h : S^{i+1} \to X$ be a map and let $F \to S^{i+1}$ be a vector bundle with $\mathrm{rank}(F) + i + 1 < \mathrm{rank}(E)$ such that F and $h^*(E)$ are stably isomorphic. Show that $S(F)$, the sphere bundle of F admits a normal structure in (X, E).

2. Let $(W; M, N)$ be a compact bordism between closed smooth simply-connected manifolds M and N and let (W, F, β) be a normal smoothing in (X, E). Show that (W, F, β) is bordant over (X, E) relative to the boundary to an $([n/2]-1)$-smoothing (W', F', β').

   Hint: The proof from lectures in the case where $M \to X$ is a homotopy equivalence can be used except for the surjetivity of $H_r(W) \to H_r(X)$: use Exercise 1 above to help.

3. Recall (Milnor-Stasheff Ch. 11) the Wu class of a closed n-manifold

   $$v(M) = 1 + v_1(M) + v_2(M) + \dots$$

   where $v_i(M) \in H^i(M; \mathbb{Z}/2)$. Two fundamental facts are as follows:

   $$x \cup v_i(M) = Sq^i(x), \quad \forall x \in H^{n-i}(M; \mathbb{Z}/2),$$

   $$w(M) = Sq(v(M)),$$

   where Sq^i is the ith Steenrod square, $Sq = 1 + Sq^1 + Sq^2 + \dots$ is the total Steenrod square and $w(M)$ is the total Stiefel-Whitney class.

   Show that a closed simply conneced 4-manifold M has even intersection form S_M if and only if $w_2(M) = 0$.

4. Let M be a closed oriented simply connected 4-manifold with odd intersection form S_M: prove that the stable normal Gauss map $\nu : M \to BSO$ is a 2-equivalence.

   Hint: recall $H^*(BSO; \mathbb{Z}/2) \cong \mathbb{Z}/2[w_2, w_3, w_4, \dots]$.

5 Thursday: Algebra

Recall that $H_+(\mathbb{Z}^k)$ denotes the hyperbolic form of rank $2k$.

1. Let $b : A \times A \to \mathbb{Z}$ be a rank $2r$ even symmetric unimodular form. Show that if b admits a Lagrangian: i.e. a rank k summand $L \subset A$ such that $b|_{L\times L} = 0$, then there is an isometry
$$\alpha : b \cong H_+(\mathbb{Z}^k)$$
such that $\alpha(L) = \mathbb{Z}^k \times \{0\}$.

2. A summand $V \subset H_+(\mathbb{Z}^k)$ is said to have a Lagrangian complement if there is a Lagrangian $L \subset H_+(\mathbb{Z}^k)$ such that $V \oplus L = H_+(\mathbb{Z}^k)$.

   Let [12] be the indicated form symmetric form over $\mathbb{Z}$ of rank 1. Find:

   (a) an embedding of [12] into $H_+(\mathbb{Z})$ with a Lagrangian complement.

   (b) an embedding of [12] into $H_+(\mathbb{Z})$ with no Lagrangian complement.

6 Friday: Surgery

Identify $S^{p+q+1} = \partial(D^{p+1} \times D^{q+1})$ so that we have the standard embedding
$$f : S^p \times D^{q+1} \hookrightarrow S^{p+q+1}$$
Given a smooth map $\alpha : S^p \to SO_{q+1}$ define the twist diffeomorphism
$$t_\alpha : S^p \times D^{q+1} \cong S^p \times D^{q+1}, \quad (x, v) \longmapsto (x, \alpha(x)v).$$
We can "re-frame" by pre-composing f with t_α to obtain the embedding
$$f_\alpha = f \circ t_\alpha : S^p \times D^{q+1} \to S^{p+q+1}.$$
Define
$$\omega(\alpha) = (S^{p+q+1} \times [0,1]) \cup_{f_\alpha} (D^{p+1} \times D^{q+1})$$
to be the *trace* of surgery on f_α and let
$$\chi(\alpha) = \partial_+(\omega(\alpha))$$
be the outcome of surgery on α.

1. Identify the diffeomorphism type of $\chi(\alpha)$ and $\omega(\alpha)$ in terms of $[\alpha] \in \pi_p(SO_{q+1})$.

2. Identify the embedding of the "dual-sphere", $D^{p+1} \times S^q \subset \chi(\alpha)$ based on your identification from the previous exercise.

3. Let $I : D^{p+q+1} \subset S^{p+q+1}$ be a standard embedding and observe that the embeddings f_α above factors through I: $f_\alpha = I \circ \bar{f}_\alpha$.

   Let M be a compact smooth $(p+q+1)$-manifold with embedding $J : D^{p+q+1} \subset M$ and consider the embedding
$$J \circ \bar{f}_\alpha : S^p \times D^{q+1} \hookrightarrow M$$
   Identify the diffeomorphism type of $\omega((J \circ \bar{f}_\alpha))$ and $\chi(J \circ \bar{f}_\alpha)$ using the first exercise.

7 Friday: Algebra

1. Let $i : [12] \hookrightarrow H_+(\mathbb{Z})$ be an embedding from Thursday, Algebra Ex 2 b which has no Lagrangian complement and let $i_0 : H_+(\mathbb{Z}) \hookrightarrow H_+(\mathbb{Z}^2)$ be some embedding. Show that the embedding
$$i \oplus i_0 : [12] \oplus H_+(\mathbb{Z}) \hookrightarrow H_+(\mathbb{Z}^3)$$
has a Lagrangian complement.

2. Let $b : A \times A \to \mathbb{Z}$ be a uni-modular symmetric bilinear form on a finitely generated free abelian group A. We define $W(\mathbb{Z}) = L^0(\mathbb{Z})$ to be the Grothendieck group of such forms modulo the sub-group with representatives containing a Lagrangian.

   Show that the signature gives a well-defined homomorphism $\text{sign} : W(\mathbb{Z}) \to \mathbb{Z}$.

   Assuming the following deep theorem:

   If b is indefinate then there exists $a \in A$ such that $a \neq 0$ and $b(a, a) = 0$,

   prove that $\text{sign} : W(\mathbb{Z}) \to \mathbb{Z}$ is an isomorphism.