SMSTC (2014/15)

Geometry and Topology

LECTURE NOTES 1: Metric and topological spaces, fundamental group and covering spaces, higher homotopy groups

Brendan Owens, Glasgow[a]

www.smstc.ac.uk

The first 4 sections of these notes (1.1 to 1.4) cover some basic point set topology, with an emphasis on examples. The last and longest section (1.5) is an introduction to the fundamental group, covering spaces, and higher homotopy groups.

1.1 Basic topology

Sections 1.1 to 1.4 are basic sketchy notes on metric and topological spaces. The first three lectures of the course will cover some of the material in these sections. Many proofs are left as exercises – for the most part these are simply a matter of applying definitions. More detail can be found in standard undergraduate point-set topology books, such as

- J. R. Munkres, *Topology: a first course*, Prentice-Hall (1975).

We will also refer to A. Hatcher, *Algebraic Topology*, Cambridge University Press (2002), freely downloadable from the author's website.

Calculus gives us the concepts of open sets in Euclidean space and continuous functions, as well as properties such as connectedness, compactness and others. In this lecture we will see how these concepts can be generalised first to *metric spaces* and then to *topological spaces*.

Notation: In these lectures $\subset$ has the same meaning as $\subseteq$, and strict inclusion is indicated by the symbol $\subsetneq$.

1.1.1 Metric spaces

The standard n-dimensional Euclidean space consists of the set

$$\mathbb{R}^n = \{(x_1, \dots, x_n) \mid x_i \in \mathbb{R}\}$$

together with the distance function $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

$$d((x_1, \dots, x_n), (y_1, \dots, y_n)) = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2}.$$

This satisfies the following properties:

(1) $d(\mathbf{x}, \mathbf{y}) \geq 0$, and $d(\mathbf{x}, \mathbf{y}) = 0 \iff \mathbf{x} = \mathbf{y}$

(2) $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$

(3) $d(\mathbf{x}, \mathbf{z}) \leq d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z})$.

More generally, any set M together with a function $d : M \times M \to \mathbb{R}$ satisfying the above three properties is called a *metric space*. The following "ϵ-δ = epsilon-delta" definition may be familiar from calculus or analysis.

[a] Brendan.Owens@glasgow.ac.uk

Definition 1.1 *Let $f : (M, d_M) \to (N, d_N)$ be a function from one metric space to another. We say f is continuous at $x \in M$ if for any $\epsilon > 0$ there exists $\delta > 0$ such that*

$$d_M(x, y) < \delta \implies d_N(f(x), f(y)) < \epsilon.$$

We say f is continuous *if it is continuous at all points $x \in M$.*

Example 1.2 *The function $f : \mathbb{R} \to \mathbb{R}$ defined by*

$$f(x) = \begin{cases} x & \text{if } x \leq 0, \\ x + 1 & \text{if } x > 0, \end{cases}$$

is continuous on $\mathbb{R} \setminus \{0\}$, but not at 0. (Verify this using ϵ-δ ideas.)

The concept of an *open set* will enable us to give a friendlier looking definition of continuity. Roughly speaking a subset U of a metric space is open if for any point x in U, all sufficiently nearby points are also in U.

Definition 1.3 *The* open ball centred at $x \in M$ with radius $r > 0$ *is the subset*

$$B(x, r) = \{y \in M \mid d(x, y) < r\} \subset M.$$

Example 1.4 (Manhattan metric) *Let $M = \mathbb{R}^2$ with*

$$d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|.$$

Sketch the unit ball $B(0, 1)$ about the origin.

Example 1.5 (Chessboard metric) *Let $M = \mathbb{R}^2$ with*

$$d((x_1, y_1), (x_2, y_2)) = \max(|x_1 - x_2|, |y_1 - y_2|).$$

Sketch the unit ball $B(0, 1)$ about the origin.

Example 1.6 (Railway metric) *Let $M = \mathbb{R}^2$ with*

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{{x_1}^2 + {y_1}^2} + \sqrt{{x_2}^2 + {y_2}^2}$$

provided $(0, 0)$, (x_1, y_1) and (x_2, y_2) are not collinear; if they are collinear, set $d((x_1, y_1), (x_2, y_2))$ to be the usual Euclidean distance. Sketch the unit ball $B(0, 1)$ about the origin.

Definition 1.7 *Let (M, d) be a metric space.*
A subset $U \subset M$ is open *if for any $x \in U$ there exists $r > 0$ with $B(x, r) \subset U$.*
A subset U is closed *if its complement $M \setminus U$ is open.*

Exercise 1.8 *Verify that a subset U of a metric space is open if and only if U is a union of open balls.*

Example 1.9 *The Euclidean, Manhattan and Chessboard metrics all give the same open sets in $\mathbb{R}^2$. The Railway metric has open sets which are not open using any of the previous three metrics. (Verify this.)*

If (M, d) is a metric space and $A \subset M$ is any subset, then A inherits the distance function d. Note that a set V in A is open if and only if $V = U \cap A$ for some open set U in M.
Let $f : M \to N$ be a function between metric spaces. For a subset $U \subset N$, the *preimage* of U under f is

$$f^{-1}(U) = \{x \in M \mid f(x) \in U\}.$$

Proposition 1.10 *Let $f : M \to N$ be a function between metric spaces. Then f is continuous if and only if the preimages of open sets are open. Equivalently, f is continuous if and only if the preimages of closed sets are closed, since $f^{-1}(N \setminus U) = M \setminus f^{-1}(U)$.*

Proof Exercise. □

1.1.2 Topological spaces

Proposition 1.11 *If (M, d) is a metric space, then*
(1) *M and $\emptyset$ are open sets;*
(2) *arbitrary unions of open sets are open;*
(3) *finite intersections of open sets are open.*

Proof Exercise. □

Note that *infinite* intersections of open sets may not be open. For example, in $\mathbb{R}$ with its usual metric,

$$\bigcap_{n=1}^{\infty}\left(-\frac{1}{n}, \frac{1}{n}\right)$$

is not open.

Definition 1.12 *A* topological space *is a set X together with a collection of subsets called* open sets *satisfying the three properties in Proposition 1.11:*
(1) X and $\emptyset$ are open sets;
(2) arbitrary unions of open sets are open;
(3) finite intersections of open sets are open.
The complements of open sets are called closed sets*.*

Examples 1.13 *Any set X can be given the* discrete *topology, in which every subset is open, or the* indiscrete *topology, in which the only open sets are X and $\emptyset$.*

Example 1.14 *Given a subset $A \subset X$ of a topological space X, the* subspace topology *on A is formed by taking $V \subset A$ to be open if and only if $V = U \cap A$ for some open set U in X.*

Proposition 1.10 now suggests the definition of continuity for functions between topological spaces.

Definition 1.15 *Let X and Y be two topological spaces. A function $f : X \to Y$ is called* continuous *(or a* continuous mapping*) if the preimage of every open set if open, or equivalently if the preimage of every closed set is closed.*

If $A \subset X$ is given the subspace topology, it is straightforward to check that a continuous function $f : X \to Y$ gives rise to a continuous restriction $f|_A : A \to Y$.
It is often convenient to specify a topology by giving a *base*; this is a collection of open sets $\mathcal{B}$ such that every open set is a union of elements of $\mathcal{B}$. For example open balls are a base for the topology in any metric space, as are open balls with rational radii.

Definition 1.16 *Two topological spaces X and Y are* homeomorphic *(written $X \cong Y$) if there exists $f : X \to Y$ which is a continuous bijection with a continuous inverse. Such an f is called a* homeomorphism*.*

This defines a notion of equivalence between topological spaces which satisfies the three properties of an equivalence relation. Also note that for a fixed space X, the set $\mathrm{Homeo}(X)$ of homeomorphisms from X to itself is a group under composition.

Examples 1.17 *Give all subsets of $\mathbb{R}^n$ the subspace topologies.*
(1) The open interval $(0, 1)$ is homeomorphic to $(0, 2)$ via $x \mapsto 2x$, and similarly any open interval (a, b) is homeomorphic to $(0, 1)$.
(2) The open interval $(-\pi/2, \pi/2)$ is homeomorphic to $\mathbb{R}$ via $x \mapsto \tan x$, and hence any open interval $(a, b) \cong \mathbb{R}$.
(3) The sets $\{0, 1, 2, 3\}$ and $\{0, 1\}$ (with any choice of topology) are not homeomorphic since there is no bijection between these sets.
(4) The sets

$$X = S^1 = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1\}$$

and $Y = (0, 1)$ are not homeomorphic: details will be given later.

Example 1.18 *Let X be any set. The* cofinite topology *is the topology in which the empty set and complements of finite sets are open. If $X = \mathbb{R}$ this is also called the* Zariski *topology.*

Example 1.19 *The Zariski topology on $\mathbb{R}^n$ (or on $\mathbf{k}^n$ for any field $\mathbf{k}$): closed sets are intersections of zero sets of polynomials in n variables. Equivalently: a base for the Zariski topology is given by complements of zero sets of polynomials.*

Examples 1.20 *Other important examples of topological spaces: S^n (the unit sphere in $\mathbb{R}^{n+1}$), orientable surfaces Σ_g, nonorientable surfaces N_g, matrix groups,...*

1.1.3 Compactness

Let X be a topological space. An *open cover* of X is a collection $\{U_\alpha\}_{\alpha\in A}$ of open subsets of X with $X = \bigcup_{\alpha\in A} U_\alpha$.
For example take $X = (0,2]$ and $U_n = (1/n, 2]$, then $\{U_n\}_{n\in\mathbb{N}}$ is an open cover of X. If we let $Y = [0,2]$ and $U_0 = [0,1/2)$ then $\{U_n\}_{n\geq 0}$ is an open cover of Y. Note that $\{U_n\}_{n\geq 100}$ also covers X; we say it is a *subcover* of $\{U_n\}_{n\in\mathbb{N}}$. Any such subcover is infinite whereas the cover $\{U_n\}_{n\geq 0}$ of Y admits the finite subcover $\{U_0, U_3\}$.

Definition 1.21 *A topological space is* compact *if every open cover admits a finite subcover.*

We recall the following fundamental theorems about compactness in Euclidean space. Note a subset of $\mathbb{R}^n$ is *bounded* if it is contained in $B(0,R)$ for some $R > 0$.

Theorem 1.22 (Heine-Borel) *Closed intervals $[a,b]$ are compact subsets of $\mathbb{R}$.*

Theorem 1.23 (Heine-Borel) *A subset U of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Proposition 1.24 *The image of a compact space under a continuous map is compact. Thus compactness is a* topological property *(preserved under homeomorphism).*

Proof Exercise. □

Thus if $X \cong Y$, then X is compact if and only if Y is compact. This shows that S^1 is not homeomorphic to $(0,1)$.
Another useful notion of compactness is defined using sequences. A sequence $x_1, x_2, \ldots$ in a topological space X converges to $x \in X$ if for any open set U containing x there exists $N \in \mathbb{N}$ with $x_n \in U$ for all $n \geq N$. A space X is *sequentially compact* if every sequence in X has a convergent subsequence. For subsets of metric spaces, sequentially compact is equivalent to compact.

1.1.4 Connectedness, path-connectedness and π_0

Definition 1.25 *A space X is* connected *if it cannot be written $X = U_1 \cup U_2$ with U_1, U_2 nonempty, open and disjoint.*

Example 1.26 *The real line $\mathbb{R}$ is connected. To see this, suppose that $\mathbb{R} = U_1 \cup U_2$ with U_1, U_2 open and disjoint and $0 \in U_1$. Let $R = \sup\{r > 0 \,|\, B(0,r) \subset U_1\}$. If R is finite then $B(0,R)$ is contained in U_1 but one of $\pm R$ is in U_2, which contradicts openness of U_2. If R is infinite then U_2 is empty.*

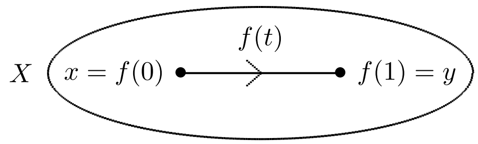
Proposition 1.27 *The continuous image of a connected set is connected.*

Proof Exercise. □

Example 1.28 *The matrix group $GL(n,\mathbb{R})$ of $n \times n$ real matrices with nonzero determinants (with subspace topology from $\mathbb{R}^{n^2}$) is disconnected, as the determinant gives a continuous surjection onto $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$, which is disconnected.*

It follows from Example 1.26 and Proposition 1.27 that the open interval $(0,1)$ is connected, since it is homeomorphic to $\mathbb{R}$. It is then an easy exercise to show that the closed interval $I=[0,1]$ is connected.

A (*continuous*) *path* in a space X is a continuous map $f:I\to X$. Two points $x,y\in X$ are said to be *joined by a path* if there is a path $f:I\to X$ with $f(0)=x$ and $f(1)=y$.



Definition 1.29 *A space X is* path-connected *if any two points in X may be joined by a path.*

Proposition 1.30 *The continuous image of a path-connected set is path-connected.*

Proof Exercise. □

Proposition 1.31 *For a topological space X,*

$$X \text{ path-connected} \implies X \text{ connected.}$$

Proof Assume X is path-connected and $X=U_1\cup U_2$ with U_1, U_2 nonempty, open and disjoint. Choose $x\in U_0$, $y\in U_1$ and a path $f:I\to X$ with $f(0)=x, f(1)=y$. Connectedness of $f(I)$ is contradicted by

$$f(I)=(f(I)\cap U_0)\cup(f(I)\cap U_1).$$

□

Using Proposition 1.31 we can easily show that many examples such as $\mathbb{R}^n$, S^n, T^2 etc. are connected, since it is fairly easy to see that they are path-connected. For example two points $x,y\in\mathbb{R}^n$ are joined by the path $f(s)=x+s(y-x)$.

Example 1.32 *The circle S^1 is not homeomorphic to the closed interval* $[0,1]$.

Proof Suppose $f:[0,1]\to S^1$ is a homeomorphism. Let $y\in(0,1)$. Then $[0,1]\setminus\{y\}$ is not connected. The homeomorphism f restricts to give a homeomorphism on $[0,1]\setminus\{y\}$ whose image is $S^1\setminus\{f(y)\}$ which is path-connected, giving a contradiction. □

Note that connectedness does not always imply path-connectedness as the following example shows.

Example 1.33 *The subspace*

$$X=\{(0,y)\,|\,y\in\mathbb{R}\}\cup\{(x,\sin(1/x))\,|\,x>0\}\subset\mathbb{R}^2$$

is connected but not path-connected.

Given two paths $f,g:I\to X$ with $f(1)=g(0)$, their *composition* or *concatenation* is the path $f\cdot g:I\to X$ given by

$$f\cdot g(s)=\begin{cases} f(2s) & \text{if } 0\le s\le 1/2,\\ g(2s-1) & \text{if } 1/2\le s\le 1.\end{cases}$$

(Continuity of $f\cdot g$ follows from the Gluing Lemma which is proved in the exercises.) We note that this is *not* a composition of functions – it is a new use of the word composition.

We can now define a relation on the points of a space X by $x\sim y$ if x and y are joined by a path.

Claim 1 *This is an equivalence relation.*

Proof Exercise. □

Equivalence classes are called *path components*. The set of path components of X is denoted $\pi_0(X)$. We will see that π_0 is a *functor* from the category **TOP** of topological spaces and continuous maps to the category **SET** of sets with functions. What this means is the following: each continuous map $f:X\to Y$

induces a function $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ given by $[x] \mapsto [f(x)]$ with the following properties:
(1) $\pi_0(\mathrm{Id}_X) = \mathrm{Id}_{\pi_0(X)}$;
(2) if $f : X \to Y$ and $g : Y \to Z$ are continuous then

$$\pi_0(g \circ f) = \pi_0(g) \circ \pi_0(f).$$

This *functoriality* is a very useful property. In particular it shows that the cardinality of π_0 is a *topological invariant.*

Proposition 1.34 *If $f : X \to Y$ is a homeomorphism then $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is a bijection.*

Proof Exercise. □

Example 1.35 *Let $X = \mathbb{R}$, $Y = \mathbb{R} \setminus \{0\}$, $Z = \{0, 1\} \subset \mathbb{R}$. Then $\pi_0(X) = \{[0]\}$, $\pi_0(Y) = \{[-1], [1]\}$, $\pi_0(Z) = \{[0], [1]\}$, This shows that X is not homeomorphic to either Y or Z. The spaces Y and Z have bijective π_0 so we need to use other means to show that they are not homeomorphic (for example there is no bijection between them).*

The converse of Proposition 1.31 is not true in general, but the following result is often useful.

Definition 1.36 *A topological space X is* locally path-connected *if for every $x \in X$ there is a path-connected open set $U \subset X$ with $x \in U$.*

Proposition 1.37 *For a topological space X,*

$$X \text{ connected and locally path-connected} \implies X \text{ path-connected.}$$

Proof The basic idea is to show that each path component is open, hence there can be only one by connectedness. □

For example, manifolds are locally path-connected, hence a connected manifold is also path-connected.

1.1.5 Hausdorff spaces

For topological spaces which are not metric spaces, there are various *separability conditions* that have important consequences. The Hausdorff condition is one of the most commonly encountered.

Definition 1.38 *A topological space X is* Hausdorff *if for every pair of distinct elements $u, v \in X$, there are disjoint open subsets $U \subset X$ and $V \subset X$ with $u \in U$ and $v \in V$.*

It is clear that every metric space is Hausdorff. To see an example of a non-Hausdorff space, consider the set with 2 elements $X = \{a, b\}$ and open sets

$$\emptyset,\ \{a\},\ X.$$

Then X is not Hausdorff with this topology. Another example is provided by taking any infinite set Y with the cofinite topology: open sets are subsets with finite complements together with $\emptyset$. The Zariski topology on $\mathbb{k}^n$ for a field $\mathbb{k}$ is also non-Hausdorff.
Here is a general fact about Hausdorff spaces.

Proposition 1.39 *Let X be a Hausdorff space. Then for each $u \in X$, the subset $\{u\} \subset X$ is closed.*

Proof For each $v \in X$ with $v \neq u$, choose disjoint open sets $U_{u,v}, V_{u,v}$ with $u \in U_{u,v}$ and $v \in V_{u,v}$. Then

$$X \setminus \{u\} = \bigcup_{v \in X \setminus \{u\}} V_{u,v},$$

so $X \setminus \{u\}$ is open, hence $\{u\}$ is closed. □

Proposition 1.40 *Let X be a Hausdorff space. Then every compact subset of X is closed.*

Proof Exercise. □

Finally, here is an important result heavily used in algebraic topology.

Proposition 1.41 *Let X be a compact space and let Y be a Hausdorff space. Then every continuous bijection $f\colon X \to Y$ is a homeomorphism.*

Proof Exercise: the proof involves showing the inverse f^{-1} is continuous. □

There are other separation properties which are commonly encountered such as normality.

1.1.6 One-point compactifications

Compactness is a good property for a topological space to possess, but most spaces are not compact. There are various ways to enlarge a space to obtain a compact space in which it sits as a "dense" subspace. The (*Alexandroff*) *one-point compactification* is particularly important.

Let X be a topological space. Let $X^* = X \cup \{\infty\}$ where ∞ is supposed to be a new point lying outside of X. Define the open sets of a topology on X^* by taking all open subsets $U \subset X$ and all sets $W \subset X^*$ containing ∞ for which $X \cap W \subset X$ is open and $X \setminus W \subset X$ is compact.

Proposition 1.42 *X^* with the above notion of open sets is a compact topological space. X^* is Hausdorff if and only if X is Hausdorff and locally compact (meaning every point has a compact neighbourhood).*

Proof Exercise. □

Here is an important family of examples.

Example 1.43 *Consider $X = \mathbb{R}^n$ with its usual metric topology. Then the one-point compactification $X^* = (\mathbb{R}^n)^*$ is homeomorphic to the unit n-sphere $S^n \subset \mathbb{R}^{n+1}$.*

One explicit homeomorphism can be produced using stereographic projection of S^n with a point deleted to $\mathbb{R}^n$. For $\mathbb{R}$ it is intuitively clear that the circle S^1 is obtained by adjoining a single "point at infinity", but this construction requires care to make the topology precise.

1.1.7 The product topology

Let X and Y be topological spaces. Thinking of them as sets, we can take the Cartesian product

$$X \times Y = \{(x, y) \,|\, x \in X,\ y \in Y\}.$$

There is a natural choice of topology on such a product which is defined so as to make the projection maps onto each factor continuous.

Product topology on $X \times Y$: a subset $U \subset X \times Y$ is open if and only if for each $(x, y) \in U$ there exist open sets $A \subset X$ and $B \subset Y$ with $(x, y) \in A \times B \subset U$.

Note that *open rectangles* $A \times B$ for open sets $A \subset X$ and $B \subset Y$ form a base for the product topology on $X \times Y$: every open set is a union of open rectangles.

Example 1.44 *The product topology on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$: a set U is open if each point in U is contained in an open rectangle contained in U. This gives the same open sets, and hence the same topology, as the Euclidean metric (compare Example 1.9).*

Example 1.45 *The* 2-dimensional *torus $T^2 = S^1 \times S^1$ is the product of two circles, with the product topology. Convince yourself that this is homeomorphic to the surface of a ring doughnut (with the subspace topology induced from $\mathbb{R}^3$).*

The product topology can also be defined for an arbitrary product

$$\prod_{\lambda \in \Lambda} X_\lambda$$

of topological spaces X_λ indexed on a set Λ. An important result about products of compact spaces is *Tychonoff's Theorem* which is actually logically equivalent to the *Axiom of Choice*:

Theorem 1.46 *For a collection of compact topological spaces X_λ $(\lambda \in \Lambda)$, the product $\prod_{\lambda\in\Lambda} X_\lambda$ is compact.*

It is a good exercise to prove this for a product of two compact spaces.
Product spaces have the following important property.

Proposition 1.47 *Let X, Y be two topological spaces and let the projection functions be $p : X \times Y \to X$ and $q : X \times Y \to Y$. Suppose that $f : Z \to X \times Y$ is a function where Z is a topological space. Then f is continuous if and only if each of the compositions $p \circ f : Z \to X$ and $q \circ f : Z \to Y$ is continuous.*

Proof Exercise. □

1.1.8 The quotient topology

Let X be a topological space and let $\sim$ be an equivalence relation on X. Then thinking of X as a set, we can form the set $X/\sim$ of equivalence classes. We choose a topology on this set so that the projection map

$$p : X \to X/\sim; \quad x \mapsto [x]$$

is continuous.

Quotient topology on $X/\sim$: a subset $U \subset X/\sim$ is open if and only if $p^{-1}(U)$ is open in X.
We call $X/\sim$ with this topology a quotient space. One simple example of a quotient space is the wedge sum of two spaces.

Definition 1.48 *Let X and Y be two spaces with chosen basepoints $x_0 \in X$ and $y_0 \in Y$. The* wedge sum *of X and Y, written $X \vee Y$, is the quotient of the disjoint union $X \sqcup Y$ by the equivalence relation given by $x_0 \sim y_0$ (all other equivalence classes consist of just one point). For example, the wedge sum $S^1 \vee S^1$ of two circles is a "figure 8".*

Another basic example is to take X to be the closed interval $[0, 1]$ and then to set $0 \sim 1$ (all other equivalence classes consist of just one point). Intuitively, you should think of this as taking the interval and bending it so that you can join one end to the other. This should lead you to expect that

$$[0, 1]/\{0 \sim 1 \cong S^1\},$$

in other words the glued up interval is a circle. To check this kind of thing carefully we need to develop some tools.

Definition 1.49 *A quotient map $q : X \to Y$ is a surjective map with the property that*

$$U \subset Y \text{ is open} \iff q^{-1}(U) \subset X \text{ is open.}$$

There are various convenient ways to recognise quotient maps. A function $f : X \to Y$ between spaces is called *open* if it takes open sets to open sets, and it is called *closed* if it takes closed sets to closed sets.

Proposition 1.50 *Let $f : X \to Y$ be surjective, continuous and either open or closed. Then f is a quotient map.*

Proof Exercise. □

Proposition 1.51 *Let X be compact and Y Hausdorff, and suppose that $f : X \to Y$ is surjective and continuous. Then f is a quotient map.*

Proof Let $U \subset X$ be closed. Then since a closed subset of a compact set is compact (exercise), and a continuous image of compact is compact, and a compact subset of Hausdorff is closed (exercise), we have $f(U)$ is closed in Y. Thus f is a quotient map by Proposition 1.50. □

Theorem 1.52 *Let $\sim$ be an equivalence relation on a space X and let $X/\sim$ be the quotient space. Suppose $q : X \to Z$ is a quotient map and that*

$$q(x) = q(y) \iff x \sim y.$$

Then $X/\sim$ is homeomorphic to Z.

Proof As above we let $[x] \in X/\sim$ denote the equivalence class of $x \in X$ and let $p : X \to X/\sim$ be given by $x \mapsto [x]$. Define $\bar{q} : X/\sim \to Z$ by $\bar{q}([x]) = q(x)$, so that

$$q = \bar{q} \circ p.$$

Then $\bar{q}$ is well-defined and a bijection (check!). Moreover

$$\begin{aligned} U \text{ is open in } X/\sim &\iff p^{-1}(U) \text{ is open in } X \\ &\iff q(p^{-1}(U)) = \bar{q} \circ p \circ p^{-1}(U) = \bar{q}(U) \text{ is open in } Z, \end{aligned}$$

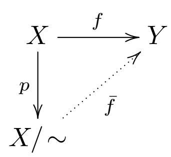
which shows that $\bar{q}$ is a homeomorphism. □

Quotient spaces have an important *universal property.*

Proposition 1.53 *Let $p : X \to X/\sim$ be the quotient mapping associated with an equivalence relation $\sim$ on the topological space X. Suppose that $f : X \to Y$ is a continuous mapping which satisfies*

$$x_1 \sim x_2 \Longrightarrow f(x_1) = f(x_2)$$

for $x_1, x_2 \in X$. Then there is a unique continuous mapping $\bar{f} : X/\sim \to Y$ such that $\bar{f} \circ p = f$.



Proof This is immediate for the underlying functions, all that needs checking is that the function $\bar{f}$ is continuous. We leave this as an exercise. □

1.1.9 Quotient space examples

We now apply Theorem 1.52 to some examples.

Example 1.54 *Let $X = [-1, 1]$, with $-1 \sim 1$ (all other equivalence classes consist of just one point). Then*

$$X/\sim \cong S^1.$$

Proof Let $Z = S^1$ and $q : X \to Z$ be given by $q(x) = e^{2\pi i x}$. Then q is a continuous surjection from a compact space to a Hausdorff space, so it is a quotient map (Proposition 1.51), and

$$q(x) = q(y) \iff x \sim y.$$

By Theorem 1.52, q descends to a homeomorphism between $X/\sim$ and S^1. □

Example 1.55 *Let $X = [0, 1] \times [0, 1]$ with equivalence relation given by*

$$\begin{aligned} (s, 0) &\sim (s, 1) \\ (0, t) &\sim (1, t). \end{aligned}$$

(Implicitly, all other equivalence classes consist of just one point.) Then $X/\sim$ is homeomorphic to the two-dimensional torus $T^2 \cong S^1 \times S^1$.

Proof Use the quotient map $q(s, t) = (e^{2\pi i s}, e^{2\pi i t})$ and Theorem 1.52. □

Examples 1.56 *There are some other interesting quotients of* $X = [0,1] \times [0,1]$.

(a) *If we take the equivalence relation given by*

$$\begin{aligned}(s,0) &\sim (s,1)\\(0,t) &\sim (1,1-t),\end{aligned}$$

then $X/\sim$ *is called the* Klein bottle.

(b) *If we take the equivalence relation given by*

$$\begin{aligned}(s,0) &\sim (1-s,1)\\(0,t) &\sim (1,1-t),\end{aligned}$$

the resulting quotient $X/\sim$ *is called the* real projective plane.

Example 1.57 *Let* $X = D_1 \sqcup D_2$ *be a disjoint union of two copies of the closed two-dimensional disk, and choose any homeomorphism*

$$\phi : \partial D_1 \to \partial D_2$$

from the boundary circle of D_1 *to the boundary circle of* D_2. *Then the quotient*

$$X/\, x \sim \phi(x) \quad \text{if } x \in \partial D_1$$

is homeomorphic to S^2.

Proof Parametrise D_1 using Cartesian coordinates r, θ, and then parametrise D_2 using polar coordinates $\rho, \phi(\theta)$. We have the quotient map

$$\begin{aligned}q : D_1 \sqcup D_2 &\to S^2\\(r,\theta) &\mapsto \begin{cases}(0,0,1) & \text{if } r = 0,\\(\cos(\theta)\sin(\pi r/2), \sin(\theta)\sin(\pi r/2), \cos(\pi r/2)) & \text{if } r \neq 0,\end{cases}\\(\rho,\phi(\theta)) &\mapsto \begin{cases}(0,0,-1) & \text{if } \rho = 0,\\(\cos(\theta)\sin(\pi\rho/2), \sin(\theta)\sin(\pi\rho/2), -\cos(\pi\rho/2)) & \text{if } \rho \neq 0.\end{cases}\end{aligned}$$

This is a continuous surjection (this uses the gluing lemma) from compact to Hausdorff so it's a quotient map, and preimages of points under q correspond to equivalence classes under $\sim$, as required. □

Example 1.57 generalises as follows:

Example 1.58 *Let* $X = D_1 \sqcup D_2$ *be a disjoint union of two copies of the closed* n*-dimensional disk* $D^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 \leq 1\}$, *and choose any homeomorphism*

$$\phi : \partial D_1 \to \partial D_2$$

from the boundary sphere of D_1 *to the boundary sphere of* D_2. *Then the quotient*

$$X/\, x \sim \phi(x) \quad \text{if } x \in \partial D_1$$

is homeomorphic to S^n.

If X is a space and $A \subset X$ is a subspace, we write X/A to denote the quotient of X by the equivalence relation under which all points of A are equivalent, and other equivalence classes consist of single points. You can imagine we crush A to a point inside X and see what is left.

Example 1.59 *Let* $D^2 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ *and* $S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. *Then*

$$D^2/S^1 \cong S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

Proof Intuitively think of what happens if you wrap a spherical ball up using a disk of elastic material, in such a way that the disk stretches around and its boundary circle is gathered together at a point at the bottom of the ball. The following map (and that of Example 1.54) is based on this intuition:

$$q(x,y) = \begin{cases} (0,0,1) & \text{if } (x,y) = (0,0), \\ \left(\dfrac{x\sin(\pi r)}{r}, \dfrac{y\sin(\pi r)}{r}, \cos(\pi r)\right) & \text{if } (x,y) \neq (0,0), \end{cases}$$

where $r = \sqrt{x^2+y^2}$. As usual this is a continuous surjection (check!) from a compact space to a Hausdorff space so it is a quotient map, and the claimed homeomorphism follows from Theorem 1.52. □

Note that Example 1.59 is a two-dimensional version of the one-dimensional Example 1.54. This works in higher dimensions also:

Example 1.60 *Let* $X = D^n = \{(x_1,\dots,x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 \leq 1\}$ *be the closed* n*-disk and* $A = S^{n-1} = \{(x_1,\dots,x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 = 1\}$ *the* $(n-1)$*-dimensional sphere. Then*

$$X/A \cong S^n = \{(x_1,\dots,x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = 1\}.$$

Proof Exercise. □

1.2 Projective spaces

A very important and interesting topological space is given by the set of lines through the origin in $\mathbb{R}^{n+1}$ for each $n \geq 0$. This is called *n-dimensional real projective space*, denoted $\mathbb{RP}^n$. To topologise $\mathbb{RP}^n$ we think of it as a quotient of $\mathbb{R}^{n+1} \setminus \{0\}$ by $\mathbf{x} \sim \mathbf{y}$ if $\mathbf{x} = \lambda\mathbf{y}$ for $\lambda \in \mathbb{R}^*$ (i.e. if $\mathbf{x}$ and $\mathbf{y}$ are on the same line through the origin). Thus we have a quotient map

$$\begin{aligned} q : \mathbb{R}^{n+1} \setminus \{0\} &\to \mathbb{RP}^n \\ \mathbf{x} &\mapsto \mathbb{R}\mathbf{x} = \text{ the line through } \mathbf{x}. \end{aligned} \tag{1.1}$$

We can give other useful descriptions of projective space. It is helpful to first establish that this space in Hausdorff:

Lemma 1.61 *Projective space is Hausdorff.*

Proof Perhaps the easiest way to see this is to note that every metric space is Hausdorff, and that the formula

$$d(L,M) = \min\{d(\mathbf{x},\mathbf{y}) : \mathbf{x} \in L, |\mathbf{x}| = 1, \mathbf{y} \in M, |\mathbf{y}| = 1\},$$

where L and M are lines through the origin in $\mathbb{R}^{n+1}$, gives a metric on $\mathbb{RP}^n$. □

Every line through the origin in $\mathbb{R}^{n+1}$ passes through the unit sphere S^n in $\mathbb{R}^{n+1}$; the same is true of a closed hemisphere D^n. Lines through the origin intersect D^n in a unique point except those which intersect its boundary ∂D^n in a pair $\pm\mathbf{x}$. Thus using Theorem 1.52 we have

$$\mathbb{RP}^n \cong (\mathbb{R}^{n+1} \setminus \{0\}) / \, \mathbf{x} \sim \lambda\mathbf{x} \tag{1.2}$$
$$\cong S^n / \, \mathbf{x} \sim -\mathbf{x} \tag{1.3}$$
$$\cong D^n / \, \mathbf{x} \sim -\mathbf{x} \text{ if } x \in \partial D^n = S^{n-1}. \tag{1.4}$$

Exercise 1.62 *Verify the homeomorphisms in lines* (1.3) *and* (1.4)*. (Use the fact that a continuous bijection from a compact space to a Hausdorff space is a homeomorphism.)*

The last two descriptions indicate why $\mathbb{RP}^n$ is called "n-dimensional" – it is an example of an *n-dimensional manifold* – in other words it is locally homeomorphic to $\mathbb{R}^n$ (as well as being a second countable Hausdorff space). You will learn more about manifolds later.

Examples 1.63 *One dimensional projective space* $\mathbb{RP}^1$ *is homeomorphic to the circle* S^1 *using* (1.4) *and Example 1.54. The real projective plane* $\mathbb{RP}^2$ *is the quotient of the disk* D^2 *by identifying opposite points on the boundary (compare Example 1.56(b)).*

Exercise 1.64 *Convince yourself that you can cut a disk out of $\mathbb{RP}^2$ in such a way that what is left is a Möbius band.*

Comparing (1.3) and (1.4) we see that $\mathbb{RP}^n$ is the union of an open disk $(D^n)^\circ$ and $\mathbb{RP}^{n-1}$. Hence by induction

$$\mathbb{RP}^n = (D^n)^\circ \cup (D^{n-1})^\circ \cup \cdots \cup (D^1)^\circ \cup \{\text{a point}\} \tag{1.5}$$

$$\cong \mathbb{R}^n \cup \mathbb{R}^{n-1} \cup \cdots \cup \mathbb{R} \cup \{\text{a point}\}. \tag{1.6}$$

This is called a *cell decomposition* of $\mathbb{RP}^n$, of which more later. The second description (1.6) follows from the first (1.5) since $(D^n)^\circ$ is homeomorphic to $\mathbb{R}^n$ for each n, but we can also see it from a slightly different point of view.

We have $\mathbb{R}^n \setminus \{0\}$ embedded in $\mathbb{R}^{n+1} \setminus \{0\}$ as the set of points with vanishing last coordinate. Applying the quotient map q from (1.1) yields

$$\mathbb{RP}^{n-1} \subset \mathbb{RP}^n.$$

The complement $\mathbb{RP}^n \setminus \mathbb{RP}^{n-1}$ is naturally homeomorphic to $\mathbb{R}^n$: to see this note that each line in $\mathbb{R}^{n+1} \setminus \{0\}$ which is not contained in $\mathbb{R}^n \setminus \{0\}$ passes through the hyperplane $x_{n+1} = 1$ exactly once, and this hyperplane is just another copy of $\mathbb{R}^n$. Thus

$$\mathbb{RP}^n \cong \mathbb{R}^n \cup \mathbb{RP}^{n-1},$$

and induction yields (1.6).

This point of view gives us another way to see that the real projective line $\mathbb{RP}^1$ is homeomorphic to S^1.

Example 1.65 *The real projective line $\mathbb{RP}^1$ is obtained from the union $D_1 \sqcup D_2$ of two disjoint closed intervals, glued together at the ends (i.e. via a homeomorphism $\partial D_1 \to \partial D_2$).*

Proof We let $D_1 = \{(x, 1) : -1 \leq x \leq 1\} \subset \mathbb{R}^2$ and $D_2 = \{(1, y) : -1 \leq y \leq 1\} \subset \mathbb{R}^2$. Every line through the origin passes through $D_1 \cup D_2$, and all but the line $y = -x$ intersect this set uniquely. The subspace $D_1 \cup D_2 \subset \mathbb{R}^2$ is homeomorphic to the disjoint union of D_1 and D_2 with one end of D_1 identified to one end of D_2, and we get $\mathbb{RP}^1$ by further identifying their other ends. (To see that this quotient is homeomorphic to $\mathbb{RP}^1$ note that the restriction of the quotient map $q : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^1$ to $D_1 \cup D_2 \subset \mathbb{R}^2$ is a continuous surjection from a compact space to a Hausdorff space; now apply Theorem 1.52.) □

We can also consider *complex projective spaces*, where we take the set of complex lines through the origin in $\mathbb{C}^{n+1}$. Thus

$$\mathbb{CP}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbf{z} \sim \lambda\mathbf{z}, \quad \lambda \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}.$$

This can be shown to be Hausdorff as in Lemma 1.61. Again we get a full set of representatives if we restrict to the unit sphere in $\mathbb{C}^{n+1}$, but each complex line intersects this in a circle (multiplying a unit vector in $\mathbb{C}^{n+1}$ by $e^{i\theta} \in S^1 \subset \mathbb{C}^*$ gives another unit vector in the same complex line). Thus (arguing as in Exercise 1.62) we have

$$\mathbb{CP}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\,\mathbf{z} \sim \lambda\mathbf{z}, \quad \lambda \in \mathbb{C}^* \tag{1.7}$$

$$\cong S^{2n-1}/\,\mathbf{z} \sim \lambda\mathbf{z}, \quad \lambda \in S^1. \tag{1.8}$$

We get a cell decomposition analagous to (1.6) above, and in exactly the same way: the inclusion of $\mathbb{C}^n \setminus \{0\}$ in $\mathbb{C}^{n+1} \setminus \{0\}$ as the set of points with $z_{n+1} = 0$ yields $\mathbb{CP}^{n-1} \subset \mathbb{CP}^n$. Each line in the complement intersects the hyperplane $z_{n+1} = 1$ in a point, so we have

$$\mathbb{CP}^n \cong \mathbb{C}^n \cup \mathbb{CP}^{n-1},$$

and by induction

$$\mathbb{CP}^n \cong \mathbb{C}^n \cup \mathbb{C}^{n-1} \cup \cdots \cup \mathbb{C} \cup \{\text{a point}\}.$$

This gives a particularly simple description of the complex projective line $\mathbb{CP}^1$ as $\mathbb{C} \cup \{\text{a point}\}$. Careful analysis of the topology shows this is in fact the one-point compactification of $\mathbb{C}$ and thus in fact $\mathbb{CP}^1$ is homeomorphic to S^2 (see Section 1.1.8). You can see this more easily by mimicking the description of $\mathbb{RP}^1$ in Example 1.65, as in the following exercise.

Exercise 1.66 *The complex projective line $\mathbb{CP}^1$ is homeomorphic to the union of two two-dimensional disks glued together along their boundary, and thus by Example 1.57 we have*

$$\mathbb{CP}^1 \cong S^2.$$

1.2.1 The Hopf map

Consideration of the description (1.8) of $\mathbb{CP}^n$ as a quotient of a sphere together with Exercise 1.66 shows we have a very interesting relationship between the low-dimensional spheres S^1, S^2 and S^3. That is to say, the quotient map q from $\mathbb{C}^2 \setminus \{0\}$ to $\mathbb{CP}^1 \cong S^2$ restricts to a map from S^3 to S^2 such that each fibre (preimage of a point) is a copy of S^1. This map was discovered by Heinz Hopf in 1931 and is therefore known as the Hopf map. It has a lot of significance in algebraic topology. This is an example of a *fibre bundle* which is an object you may meet again later in the course; it is often denoted

$$S^1 \to S^3 \to S^2.$$

The real projective line gives an analogous but less interesting map from S^1 to S^1 and a fibre bundle

$$S^0 \to S^1 \to S^1.$$

One can also form projective spaces using the *quaternions* (isomorphic to $\mathbb{R}^4$ as a real vector space, but with a multiplication, just as $\mathbb{C}$ is isomorphic to $\mathbb{R}^2$) or the *octonions* (isomorphic to $\mathbb{R}^8$ as a real vector space). The projective lines in each case give rise to fibre bundles

$$\begin{aligned} &S^3 \to S^7 \to S^4, \quad \text{and} \\ &S^7 \to S^{15} \to S^8. \end{aligned}$$

A theorem of Frank Adams from 1960 tells us these are the only possible fibre bundles of the form $S^m \to S^{m+n} \to S^n$.

1.3 CW complexes

We give a brief introduction to another important construction which uses the quotient topology.

Let X be a topological space, and let $\phi : S^{n-1} \to X$ be a continuous map. We say

$$X' = X \sqcup D^n / \, x \sim \phi(x)$$

is obtained from X by *attaching an n-cell.*

For example we can build S^1 by attaching a 1-cell to a single point. (Or by adding two 1-cells to a discrete space consisting of two points.) We can also build a torus by starting with a point, attaching two 1-cells, and then attaching a 2-cell. We have also seen that $\mathbb{RP}^n$ can be formed by starting with a point and then attaching one cell in each dimension up to n (similarly for $\mathbb{CP}^n$ with a cell in each even dimension). More generally:

Definition 1.67 *A* CW complex *is a space built from a discrete set by successively adding higher dimensional cells*

$$X^0 \subset X^1 \subset X^2 \subset \cdots \subset X.$$

That is to say, we start with a discrete set X^0. (The points of X^0 are called 0-cells.) We form X^n by attaching n-cells to X^{n-1}. Either $X = X^n$ for some n or else $X = \cup_{n=1}^{\infty} X^n$. In the latter case we use the "weak topology":

$$A \subset X \text{ is open} \iff A \cap X^n \text{ is open in } X^n \text{ for all } n.$$

This might sound like an elaborate construction but in fact just about every example we have seen in these lectures can be built in this way. It turns out CW complexes are a useful set to prove theorems about, especially in algebraic topology.

A one dimensional CW complex (formed by attaching 1-cells to a discrete set) is called a *graph.*

1.4 Homotopy and homotopy equivalence

We will now consider an equivalence relation on topological spaces which is weaker than homeomorphism; we will see later that this gives rise to isomorphisms of many invariants in algebraic topology. We will also meet along the way an equivalence class on the set of continuous maps from one space to another.

We begin with a special kind of continuous map called a *deformation retraction.*

Definition 1.68 *Let A be a subspace of a topological space X. A deformation retraction of X onto A is a continuous*

$$F : X \times I \to X$$

satisfying

$$F(x, 0) = x, \quad F(a, t) = a, \quad F(x, 1) \in A,$$

for all $x \in X$, $a \in A$, $t \in I$.

If a deformation retraction of X onto A exists, we say that A is a *deformation retract* of X.

Example 1.69 *Let X be the* annulus $S^1 \times [0, 1]$ *and let $A = S^1 \times \{0\} \subset X$. The map*

$$\begin{aligned} F : X \times I &\to X \\ ((e^{i\theta}, s), t) &\mapsto (e^{i\theta}, st) \end{aligned}$$

is a deformation retraction of X onto A.

Example 1.70 *Let X be the* Möbius strip $[-1, 1] \times [0, 1]/(r, 0) \sim (-r, 1)$ *and let A be the core circle $\{0\} \times [0, 1]/\sim$. The map*

$$\begin{aligned} F : X \times I &\to X \\ ((r, s), t) &\mapsto (rt, s) \end{aligned}$$

is a deformation retraction of X onto A. (Note that F is well-defined since

$$F((r, 0), t) = (rt, 0) \sim (-rt, 1) = F((-r, 1), t).)$$

For many examples of deformation retractions it is difficult to write down an explicit formula as we have done in Examples 1.69 and 1.70. In some simple cases we may illustrate a deformation retraction by drawing a picture of X together with arrows with endpoints in A showing the paths

$$t \mapsto F(x, t).$$

For example the diagrams in Figure 1.1 represent the deformation retractions F from Examples 1.69 and 1.70. Note that each $x \in X$ is on a unique arrow, and the path $t \mapsto F(x, t)$ is the constant-speed path along that arrow from x to the endpoint of the arrow in A.

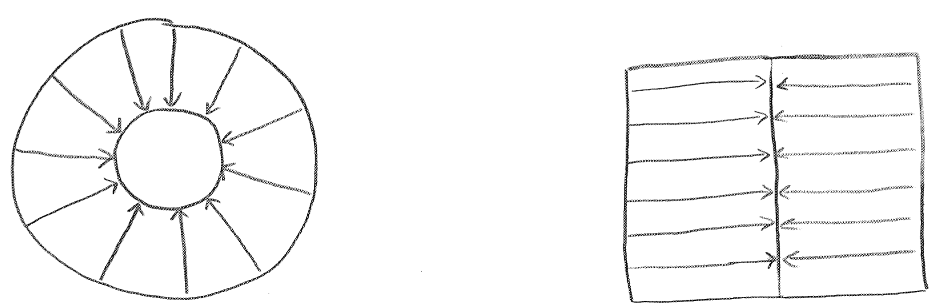


Figure 1.1: Illustrating deformation retractions.

Care must be taken so that the diagram actually describes a continuous F; for example the diagram in Figure 1.2 is an attempt to construct a deformation retraction from S^1 to the point -1. Looking closely however we see that the function $F : S^1 \times I \to S^1$ that it defines is not continuous at 1 (think about this).

We will give another important example of a deformation retraction but first we will introduce some terminology. A *pointed topological space* is a pair (X, x_0) where X is a space and $x_0 \in X$. Suppose that (X, x_0) and (Y, y_0) are pointed topological spaces. Their *wedge sum* $X \vee Y$ is the pointed space given by the quotient $X \sqcup Y / x_0 \sim y_0$, with the basepoint $[x_0] = [y_0]$. For example $S^1 \vee S^1$ is homeomorphic to the number 8 (there is no need to specify basepoints since the result is the same for any choice of x_0, y_0).

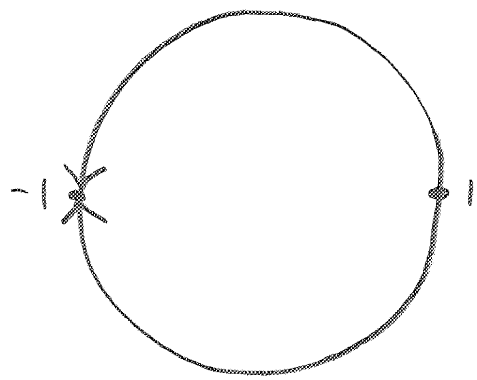


Figure 1.2: *Not* a deformation retraction of S^1 onto a point.

Example 1.71 *There exists a deformation retraction of the complement of a point in the torus*

$$T^2 = [0,1] \times [0,1] / \begin{array}{l} (0,t) \sim (1,t) \\ (s,0) \sim (s,1). \end{array}$$

onto $S^1 \vee S^1$.

Proof Let A be the subspace of the torus given by the four sides of the unit square, quotiented by $\sim$, and note that $A \cong S^1 \vee S^1$. (The top and bottom sides give one circle and the left and right sides give another, and the 4 corners of the square give the point at which the two circles are glued together.) Let

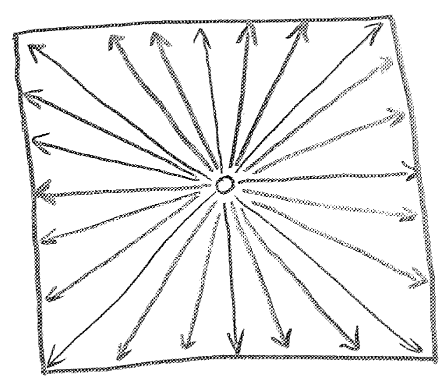


Figure 1.3: The punctured torus deformation retracts onto $S^1 \vee S^1$.

X be the complement of the point $(1/2, 1/2)$ in T^2. A deformation retraction of X onto A is illustrated in Figure 1.3. (You can write down a formula for this if you like.) □

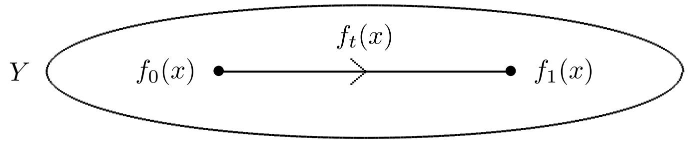
From the definition it is clear that deformation retraction does not immediately give an equivalence relation on topological spaces: if A is a deformation retract of X, it will not typically be the case that X is a deformation retract of A. However we can consider the equivalence relation this generates. We start with an equivalence relation on functions.

Definition 1.72 *Two continuous maps* $f_0, f_1 : X \to Y$ *are* homotopic*, written* $f_0 \simeq f_1$ *if there exists a continuous*

$$f : X \times I \to Y; \quad (x,t) \mapsto f(x,t) = f_t(x).$$

Such an f *is called a* homotopy *between* f_0 *and* f_1.

Think of a homotopy as a single 'take' in a film, with f_t the position of the actors at time t, starting at f_0 and ending at f_1.



Definition 1.73 *Two maps $f_0, f_1 : X \to Y$ are* homotopic relative to $A \subset X$*, or* homotopic rel A*, written $f_0 \simeq_A f_1$ or $f_0 \simeq f_1$ rel A, if there exists f as in Definition 1.72 with $f_t(a)$ constant with respect to t for all $a \in A$.*

Two paths $f_0, f_1 : I \to X$ are homotopic with fixed endpoints means they are homotopic relative to $\{0, 1\}$; however in a slight abuse of terminology one often simply says they are homotopic, and writes $f_0 \simeq f_1$ instead of $f_0 \simeq_{\{0,1\}} f_1$.

Example 1.74 *If $X = \{x\}$ is a space with one element x, a continuous map $f : X \to Y$ is the same as an element $f(x) \in Y$. A homotopy $h : f \simeq g : X \to Y$ is the same as a path $h : I \to Y$ with initial point $h(0) = f(x) \in Y$ and terminal point $h(1) = g(x) \in Y$. A homotopy $h : f \simeq f : X \to Y$ is the same as a closed path $h : I \to Y$.* $\square$ $\square$

Exercise 1.75 *Homotopy is an equivalence relation on the set of continuous functions from X to Y.*

In general, geometry is used to construct homotopies, and algebra is used to show that homotopies with certain properties cannot exist.

Definition 1.76 *Two spaces X and Y are* homotopy equivalent*, written $X \simeq Y$, if there exist maps $f : X \to Y$ and $g : Y \to X$ with $f \circ g \simeq \mathrm{Id}_Y$ and $g \circ f \simeq \mathrm{Id}_X$. The maps f and g are called* homotopy equivalences*, and also* homotopy inverses *of each other.*

Exercise 1.77 *Verify that the inclusion $f : S^n \to \mathbb{R}^{n+1} \backslash \{0\}$ is a homotopy equivalence, with homotopy inverse*

$$g \ : \ \mathbb{R}^{n+1} \backslash \{0\} \to S^n \ ; \ x \mapsto \frac{x}{\|x\|} \ .$$

Exercise 1.78 *Verify that homotopy equivalence is an equivalence relation on topological spaces.*

Exercise 1.79 *Verify that if X deformation retracts onto A then $X \simeq A$.*

Definition 1.80 *A space X is* contractible *if it is homotopy equivalent to* $\{\mathrm{pt.}\}$*.*

Exercise 1.81 *(i) A subset $X \subseteq \mathbb{R}^n$ is convex if for any $x, y \in X$ the line segment*

$$[x, y] \ = \ \{(1 - t)x + ty \,|,\ 0 \leqslant t \leqslant 1\}$$

is contained in X. Verify that such a convex set X is contractible.
(ii) The n-dimensional Euclidean space $\mathbb{R}^n$ is contractible, by (i).
(iii) The unit n-ball $D^n = \{x \in \mathbb{R}^n \,|\, \|x\| \leqslant 1\}$ is contractible, by (i).

Example 1.82 *Let X be a disk with two holes. To be precise we may take this to be the closed unit disk in $\mathbb{C}$ complement two open disks of radius $1/4$ and centres $\pm 1/2$. Then X deformation retracts onto each of $S^1 \vee S^1$, the spectacle graph (two circles joined by a line segment) and the Θ-graph (a circle union a diameter). This is illustrated in Figure 1.4.*
It follows that all 4 of these spaces are homotopy equivalent to each other.

The following theorem tells us that the method of Example 1.82 is in some sense always applicable to show that spaces are homotopy equivalent. The proof of this theorem is found in Hatcher (Corollary 0.21).

Theorem 1.83 *Two spaces X and Y are homotopy equivalent if and only if there is a third space Z which deformation retracts onto each of X and Y.*

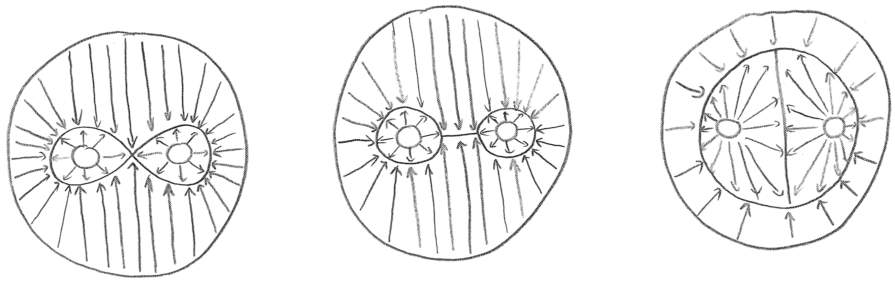


Figure 1.4: Three deformation retractions of a disk with two holes..

1.5 Fundamental group, covering spaces, higher homotopy groups

1.5.1 Books

The main reference is Allen Hatcher's downloadable book **Algebraic Topology** which is an excellent introduction to algebraic topology. Whenever possible I have included a page reference to the book, in the form [ATn].

Warning/promise: this book goes far beyond the syllabus of the SMSTC course.
Other references
1. Armstrong **Basic Topology** (Springer 1983)
2. Kirwan **Complex Algebraic Curves** (CUP 1992)
3. Stillwell **Classical Topology and Combinatorial Group Theory** (Springer 1980)
4. Stillwell **Geometry of Surfaces** (Springer 1992)
5. Zeeman **Introduction to topology** (Warwick notes 1966)
http://www.maths.ed.ac.uk/~aar/surgery/ecztop.pdf

1.5.2 Topological invariants

How does one recognize topological spaces, and distinguish between them? In the first instance, it is not even clear if the Euclidean spaces $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3, \dots$ are topologically distinct. Standard linear algebra shows that they are all non-isomorphic as vector spaces: it follows that $\mathbb{R}^m$ is diffeomorphic to $\mathbb{R}^n$ if and only if $m = n$, since the differential of a diffeomorphism is an isomorphism of vector spaces. In 1878 Cantor constructed bijections $\mathbb{R} \to \mathbb{R}^n$ for $n \geqslant 2$, which however were not continuous. In 1890 Peano constructed continuous surjections $\mathbb{R} \to \mathbb{R}^n$ for $n \geqslant 2$, the 'space-filling curves'. Thus there might also be continuous bijections with continuous inverses, i.e. homeomorphisms. It was only proved in 1910 by Brouwer that $\mathbb{R}^m$ is homeomorphic to $\mathbb{R}^n$ if and only if $m = n$.

Algebraic topology deals with topological invariants of spaces, that is functions I which associate to a topological space X an object $I(X)$ which may be either a number or an algebraic structure such as a group. The essential requirement is that homeomorphic spaces X, Y have the same invariant $I(X) = I(Y)$, where $=$ means 'isomorphic to' for algebraic invariants. Thus if X, Y are such that $I(X) \neq I(Y)$ then X, Y are not homeomorphic.
Here are some examples:

1.E.1. The dimension of a Euclidean space $\mathbb{R}^n$, $I(\mathbb{R}^n) = n$.

1.E.2. The genus $g(\Sigma)$ of an orientable surface Σ. (1850's). [AT51]

1.E.3. The Betti numbers (1860's). [AT130]

1.E.4. The fundamental group $\pi_1(X)$ (Poincaré, 1895). [AT26]

1.E.5. The homology groups $H_*(X)$ (1920's). [AT160]

1.E.6. The cohomology ring $H^*(X)$ (1930's). [AT191]

1.E.7. The higher homotopy groups $\pi_*(X)$ (1930's). [AT340]

The number of path-components in a space X (see Section 1.1.4 above)

$$|\pi_0(X)| \in \{0, 1, 2, 3, \ldots, \infty\}$$

is perhaps the simplest topological invariant: if $m \neq n$ a space with m path-components cannot be homeomorphic to a space with n path-components. By definition, a space X is path-connected if $|\pi_0(X)| = 1$, i.e. if for any $x_0, x_1 \in X$ there exists a path from x_0 to x_1.

Regard S^1 as the unit circle in the complex plane $\mathbb{C}$. A 'loop' in a space X at a point $x \in X$ is a continuous map $\omega : S^1 \to X$ such that $\omega(1) = x \in X$. The fundamental group $\pi_1(X, x)$ of X at $x \in X$ is defined geometrically to be the set of homotopy classes of loops $\omega : S^1 \to X$ at x, with the homotopies $\{\omega_t \,|\, 0 \leqslant t \leqslant 1\}$ required to be such that $\omega_t(1) = x$.
If $x_0, x_1 \in X$ are in the same path component (i.e. joined by a path) then $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic. For a path-connected space X, $\pi_1(X)$ denotes any one of the isomorphic groups $\pi_1(X, x)$ $(x \in X)$.
Here are the key properties of the fundamental group:

1.E.8. A continuous map $f : X \to Y$ induces a group morphism $f_* : \pi_1(X) \to \pi_1(Y)$ which depends only on the homotopy class of f.

1.E.9. For any space X the identity function $1_X : X \to X$ induces the identity morphism

$$(1_X)_* \;=\; 1_{\pi_1(X)} \;:\; \pi_1(X) \to \pi_1(X) \;.$$

1.E.10. For any continuous maps $f : X \to Y$, $g : Y \to Z$

$$(gf)_* \;=\; g_* f_* \;:\; \pi_1(X) \to \pi_1(Z) \;.$$

1.E.11. If f is a homotopy equivalence then f_* is an isomorphism. Thus spaces with non-isomorphic fundamental groups cannot be homotopy equivalent, and a fortiori cannot be homeomorphic.

The isomorphism class of $\pi_1(X)$ is a topological invariant of X. A space X is 'simply-connected' if it is path-connected and $\pi_1(X) = \{1\}$, i.e. every loop is homotopic to a constant loop.
In many cases it is possible to actually compute $\pi_1(X)$, and to use the fundamental group to make interesting statements about topological spaces. Here are some examples:

1.E.12. The Euclidean spaces $\mathbb{R}^n$ $(n \geqslant 1)$ are all simply-connected, with $\pi_1(\mathbb{R}^n) = \{1\}$.

1.E.13. The fundamental group of the circle S^1 is the infinite cyclic group

$$\pi_1(S^1) \;=\; \mathbb{Z} \;.$$

Every loop $\omega : S^1 \to S^1$ is homotopic to the standard loop going round S^1 d times

$$\omega_d \;:\; S^1 \to S^1 \;;\; z \mapsto z^d \;\;\text{(complex multiplication)}$$

for a unique $d \in \mathbb{Z}$ called the **degree** of ω. The function $\pi_1(S^1) \to \mathbb{Z}; \omega \mapsto \text{degree}(\omega)$ is an isomorphism. [AT29]

1.E.14. Every loop $\omega : S^1 \to \mathbb{C}\backslash\{0\}$ is homotopic to $\omega_d : S^1 \to S^1 \subset \mathbb{C}\backslash\{0\}$ for a unique $d \in \mathbb{Z}$ called the **winding number** of ω. Cauchy's theorem computes the winding number as a closed contour integral

$$\frac{1}{2\pi i} \oint_{\omega} \frac{dz}{z} \;=\; d \;.$$

1.E.15. The n-sphere S^n has $\pi_1(S^n) = \{1\}$ for $n \geqslant 2$. [AT35]

1.E.16. The n-dimensional projective space $\mathbb{RP}^n$ has $\pi_1(\mathbb{RP}^n) = \mathbb{Z}/2\mathbb{Z}$ for $n \geqslant 2$. [AT74]

1.E.17. The fundamental group of the closed orientable surface $\Sigma_g = M(g)$ of genus $g \geqslant 0$ has $2g$ generators and 1 relation

$$\pi_1(M(g)) \;=\; \langle a_1, b_1, \ldots, a_g, b_g \,|\, [a_1, b_1] \ldots [a_g, b_g] \rangle$$

with $[a, b] = a^{-1}b^{-1}ab$ the commutator of a, b. In particular, $M(0) = S^2$ is the sphere, with $\pi_1(M(0)) = \{1\}$, and $M(1) = S^1 \times S^1$ is the torus with $\pi_1(M(1)) = \mathbb{Z} \oplus \mathbb{Z}$, the free abelian group on 2 generators. Since the groups $\pi_1(M(g))$ $(g \geqslant 0)$ are all non-isomorphic, the surfaces $M(g)$ are non-homeomorphic.

[AT51]

1.E.18. If $K : S^1 \subset S^3$ is a knot then $\pi_1(S^3 \backslash K(S^1))$ is a topological invariant of the knot. For example, if $K_0 : S^1 \subset S^3$ is the trivial knot and $K_1 : S^1 \subset S^3$ is the trefoil knot then

$$\pi_1(S^3 \backslash K_0(S^1)) \;=\; \mathbb{Z} \;,\; \pi_1(S^3 \backslash K_1(S^1)) \;=\; \langle a, b \,|\, aba = bab \rangle \qquad \text{[AT55]}$$

These groups are not isomorphic (since one is abelian and the other one is not abelian), so that K_0, K_1 are essentially distinct knots. In particular, this algebra shows that the trefoil cannot be unknotted.

1.E.19. If $L : S^1 \cup \cdots \cup S^1 \subset S^3$ is a link (= knot in the case of a single S^1) then $\pi_1(S^3 \backslash L(S^1 \cup \cdots \cup S^1))$ is a topological invariant of the link. For example, if $L_0 : S^1 \cup S^1 \subset S^3$ is the trivial link then $\pi_1(S^3 \backslash L_0(S^1 \cup S^1)) = \mathbb{Z} * \mathbb{Z}$ is the free nonabelian group on 2 generators, while if $L_1 : S^1 \cup S^1 \subset S^3$ is the Hopf link then $\pi_1(S^3 \backslash L_1(S^1 \cup S^1)) = \mathbb{Z} \oplus \mathbb{Z}$. [AT24,47]

The Seifert-van Kampen Theorem states that the fundamental group of a union $X = X_1 \cup X_2$ of path-connected spaces X_1, X_2 with path-connected intersection $Y = X_1 \cap X_2$ is the amalgamated free product $\pi_1(X) = \pi_1(X_1) *_{\pi_1(Y)} \pi_1(X_2)$. [AT43]

The "figure 8" $S^1 \vee S^1$ has $\pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z}$. [AT40,77]

Every group G is the fundamental group $G = \pi_1(X)$ of some path-connected space X, and every group morphism $\phi : G \to H$ is the induced morphism $\phi = f_*$ of a continuous map $f : X \to Y$ with $\pi_1(X) = G$, $\pi_1(Y) = H$. [AT89]

Every set admits the 'discrete' topology, in which every subset is open. A 'covering' of a space X with 'fibre' a discrete space F is a continuous map $p : \widetilde{X} \to X$ such that for each $x \in X$ there exists an open subset $U \subseteq X$ with $x \in U$, and with a homeomorphism $\phi_U : F \times U \to p^{-1}(U)$ such that $p \circ \phi_U(a, u) = u \in U \subseteq X$ for all $a \in F$, $u \in U$. As a set $\widetilde{X} = X \times F$, but it is the topology on $\widetilde{X}$ which makes the covering interesting.

Let us informally call a space 'reasonable' if it is a simplicial complex or a CW complex, or a 'Δ-complex' in the sense of [AT102]. A reasonable space X which is path-connected has a 'universal covering' $p : \widetilde{X} \to X$, which is a covering with $\widetilde{X}$ simply-connected. [AT64]

There are two key results for universal covers:

1.E.20. The fibre of a universal covering $p : \widetilde{X} \to X$ is the fundamental group $\pi_1(X)$, and there is defined an isomorphism of groups

$$\pi_1(X) \;\cong\; \mathrm{Homeo}_p(\widetilde{X})$$

with $\mathrm{Homeo}_p(\widetilde{X})$ the group of homeomorphisms $h : \widetilde{X} \to \widetilde{X}$ such that $ph = p : \widetilde{X} \to X$, called the 'covering translations'.

1.E.21. For a path-connected space X with a universal covering $p : \widetilde{X} \to X$ every subgroup $G \subseteq \pi_1(X)$ determines a covering projection

$$p_G \;:\; \widetilde{X}/G \;=\; \widetilde{X}/\{x \sim y \text{ if } y = xg \text{ for some } g \in G\} \to X \;;\; x \mapsto p(x) \;.$$

The fibre of p_G is the set $[\pi_1(X); G]$ of left G-cosets $xG \subseteq \pi_1(X)$ $(x \in \pi_1(X))$, and

$$(p_G)_* \;=\; \text{inclusion} \;:\; \pi_1(\widetilde{X}/G) \;=\; G \to \pi_1(X) \;.$$

Moreover, if $q : Y \to X$ is an arbitrary covering of X with Y path-connected, then there exists a subgroup $G \subseteq \pi_1(X)$ such that $q = p_H$, $Y = \widetilde{X}_H$, and the fibre is $F = [\pi_1(X); G]$. There is a one-one correspondence between coverings $q : Y \to X$ with Y path-connected and the conjugacy classes of subgroups $G \subseteq \pi_1(X)$. By definition, two subgroups $G, G' \subseteq \pi_1(X)$ are **conjugate** if $G' = xGx^{-1}$ for some $x \in \pi_1(X)$. [AT67]

The simplest non-trivial example of a covering is:

1.E.22. The real line $\mathbb{R}$ is simply-connected and the function $p : \mathbb{R} \to S^1; t \mapsto e^{2\pi i t}$ is a universal covering, with $\text{Homeo}_p(\mathbb{R}) = \mathbb{Z}$ the infinite cyclic group generated by $\mathbb{R} \to \mathbb{R}; x \mapsto x + 1$. [AT56]

Note how much easier it is easier to compute $\text{Homeo}_p(\mathbb{R}) = \mathbb{Z}$ than $\pi_1(S^1) = \mathbb{Z}$ directly from the definition!

The 'nth homotopy group' $\pi_n(X, x)$ is defined geometrically to be the set of homotopy classes of continuous maps $\omega : S^n \to X$ such that $\omega(1) = x \in X$, just like $\pi_1(X, x)$ but for all $n \geqslant 1$. As for $n = 1$, if $x_0, x_1 \in X$ are in the same path component then $\pi_n(X, x_0)$ and $\pi_n(X, x_1)$ are isomorphic. For a path-connected space X $\pi_n(X)$ denotes any one of the isomorphic groups $\pi_n(X, x)$ $(x \in X)$.

Here are some facts about the higher homotopy groups $\pi_*(X)$:

1.E.23. For $n \geqslant 2$ $\pi_n(X)$ is abelian with a $\pi_1(X)$-action. [AT340]

1.E.24. If X is contractible (= homotopy equivalent to a point) then $\pi_*(X) = 0$, e.g. $\pi_*(\mathbb{R}^m) = 0$.

1.E.25. $\pi_n(\mathbb{R}^{m+1} \backslash \{0\}) = \pi_n(S^m) = \begin{cases} \mathbb{Z} & \text{if } n = m \\ 0 & \text{if } n < m. \end{cases}$ [AT349,361]

1.E.26. $\pi_3(S^2) = \mathbb{Z}$ (Hopf, 1926) [AT474]

1.E.27. Although the homotopy groups $\pi_n(S^m)$ for $n > m$ have been studied intensively for the last 70 years, they are still largely unknown!

1.E.28. $\pi_n(X) = \pi_{n-1}(\Omega X)$ with $\Omega X = (X, x)^{S^1}$ the space of loops in X at $x \in X$, so that for $n \geqslant 2$

$$\pi_n(X) \;=\; \pi_1(\Omega^{n-1} X)$$

with $\Omega^{n-1} X = \Omega\Omega \dots \Omega X$. [AT395]

1.E.29. A continuous map $f : X \to Y$ induces group morphisms $f_* : \pi_*(X) \to \pi_*(Y)$ such that

$$(1_X)_* \;=\; 1_{\pi_*(X)} \;:\; \pi_*(X) \to \pi_*(X) \;,\; (gf)_* \;=\; g_* f_* \;:\; \pi_*(X) \to \pi_*(Z)$$

with $g : Y \to Z$. If f is a homotopy equivalence then the f_* are isomorphisms.

1.E.30. A map of reasonable path-connected spaces $f : X \to Y$ is a homotopy equivalence if and only if the morphisms $f_* : \pi_*(X) \to \pi_*(Y)$ are isomorphisms. [AT346]

Here is a consequence of the computation

$$\pi_n(S^m) \;=\; H_n(S^m) \;=\; \begin{cases} 0 & \text{if } 0 < n < m \\ \mathbb{Z} & \text{if } n = 0 \text{ or } m \;. \end{cases}$$

If $m \neq n$ then the Euclidean spaces $\mathbb{R}^m, \mathbb{R}^n$ cannot be homeomorphic. For if there existed a homeomorphism then $\mathbb{R}^m \backslash \{0\}$ would be homeomorphic to $\mathbb{R}^n \backslash \{0\}$ and

$$\pi_{m-1}(\mathbb{R}^m \backslash \{0\}) \;=\; \mathbb{Z} \;=\; \pi_{m-1}(\mathbb{R}^n \backslash \{0\}) \;=\; 0 \;,$$

a contradiction.

1.6 The fundamental group $\pi_1(X)$

As already indicated, the construction of the fundamental group uses paths and homotopies.

Definition 1.84 *(i) A* closed path *at $x \in X$ is a path $\alpha : I \to X$ such that $\alpha(0) = \alpha(1) = x \in X$.*
(ii) A loop *at $x \in X$ is a continuous map $\omega : S^1 \to X$ such that $\omega(1) = x \in X$.*

Use the homeomorphism

$$[0,1]/(0 \sim 1) \to S^1 \ ; \ [t] \mapsto e^{2\pi i t} \ .$$

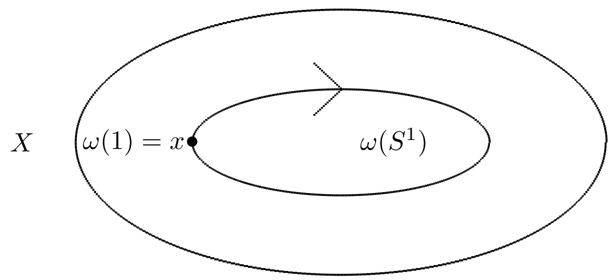
We have that a closed path $\alpha : I \to X$ at $x \in X$ is essentially the same as a loop $\omega : S^1 \to X$ at $x \in X$, with

$$\alpha(t) \ = \ \omega(e^{2\pi i t}) \in X \ .$$

Definition 1.85 *(i) A* based space (X, x) *is a space with a base point* $x \in X$*.*
(ii) A based continuous map $f : (X, x) \to (Y, y)$ *is a continuous map* $f : X \to Y$ *such that* $f(x) = y \in Y$*.*
(iii) A based homotopy $h : f \simeq g : (X, x) \to (Y, y)$ *is a homotopy* $h : f \simeq g : X \to Y$ *such that* $h(x, t) = y \in Y$ *(*$t \in I$*).*

Given based spaces (X, x) and (Y, y), based homotopy is an equivalence relation on the set of based continuous maps $f : (X, x) \to (Y, y)$.

Definition 1.86 *A* based loop *is a based continuous map* $\omega : (S^1, 1) \to (X, x)$ *where* $1 = (1, 0) \in S^1$*.*



□

Homotopy theory uses the topological properties of closed paths $I \to X$ and loops $S^1 \to X$ and the algebraic properties of groups to decide whether topological spaces are homotopy equivalent. Since I is contractible any two paths $I \to X$ with image in the same path component are homotopic. It is necessary to keep the endpoints fixed!

The fundamental group $\pi_1(X, x)$ will be defined, for any space X and point $x \in X$, to be the set of 'rel $\{0, 1\}$ homotopy classes' (Definition 1.73) of closed paths $\alpha : [0, 1] \to X$ such that

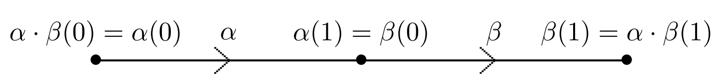
$$\alpha(0) \ = \ \alpha(1) \ = \ x \in X \ ,$$

The rel $\{0, 1\}$ homotopy classes of closed paths $\alpha : I \to X$ such that $\alpha(0) = \alpha(1) = x \in X$ are in one-one correspondence with the rel $\{1\}$ homotopy classes of loops $\omega : S^1 \to X$ with $\omega(1) = x \in X$.

Recall that the *concatenation* of paths $\alpha : I \to X$, $\beta : I \to X$ with $\alpha(1) = \beta(0) \in X$ is the path

$$\alpha \cdot \beta \ : \ I \to X \ ; \ t \mapsto \begin{cases} \alpha(2t) & \text{if } 0 \leqslant t \leqslant 1/2 \\ \beta(2t - 1) & \text{if } 1/2 \leqslant t \leqslant 1 \end{cases}$$

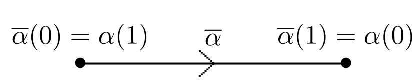
which starts at $\alpha(0)$, follows along α at twice the speed in the first half, switching at $\alpha(1) = \beta(0)$ (at half-time) to follow β at twice the speed in the second half.



The *reverse* of a path $\alpha : I \to X$ is the path

$$\overline{\alpha} \ : \ I \to X \ ; \ t \mapsto \alpha(1 - t)$$

retracing α, with



Definition 1.87 *The* fundamental group $\pi_1(X,x)$ *is the set of rel* $\{0,1\}$ *homotopy classes* $[\alpha]$ *of closed paths* $\alpha : I \to X$ *such that*

$$\alpha(0) \ = \ \alpha(1) \ = \ x \in X$$

with group law

$$\pi_1(X,x) \times \pi_1(X,x) \to \pi_1(X,x) \ ; \ ([\alpha],[\beta]) \mapsto [\alpha][\beta] = [\alpha \cdot \beta] \ ,$$

inversion by

$$\pi_1(X,x) \to \pi_1(X,x) \ ; \ [\alpha] \mapsto [\alpha]^{-1} \ = \ [\overline{\alpha}]$$

and neutral element $[e_x] \in \pi_1(X,x)$ *the class of the constant path*

$$e_x \ : \ I \to X \ ; \ t \mapsto x \ .$$

It is of course also possible to regard $\pi_1(X,x)$ as the set of rel $\{1\}$ homotopy classes $[\omega]$ of loops $\omega : S^1 \to X$ such that $\omega(1) = x \in X$. The path formulation is more convenient for algebra, while the loops are more geometric.

Theorem 1.88 *The fundamental group* $\pi_1(X,x)$ *is a group.*

Proof that $[\alpha][e_x] = [\alpha] \in \pi_1(X,x)$.
Define a rel $\{0,1\}$ homotopy

$$h \ : \ \alpha \cdot e_x \simeq \alpha \ : \ I \to X$$

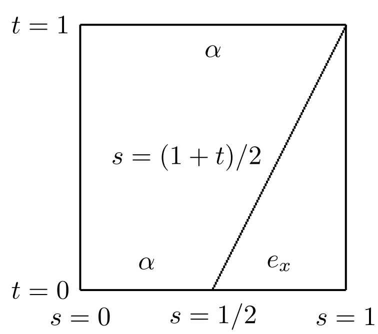
by

$$h \ : \ I \times I \to X \ ; \ (s,t) \mapsto \begin{cases} \alpha(2s/(1+t)) & \text{if } s \leqslant (1+t)/2 \\ p & \text{if } s \geqslant (1+t)/2 \ . \end{cases}$$

To make sense of this formula draw the unit square in the (s,t)-plane and join the point $(1/2,0)$ to the point $(1,1)$ by the line $s = (1+t)/2$. Think what happens at each time $t \in I$: the continuous map

$$h_t \ : \ I \to X \ ; \ s \mapsto h_t(s) = h(s,t)$$

starts by going along α at $2/(1+t)$ the speed on $[0,(1+t)/2]$, and then stays put at x on $[(1+t)/2,1]$. The homotopy h starts at $h_0 = \alpha \cdot e_x$ and ends at $h_1 = \alpha$.

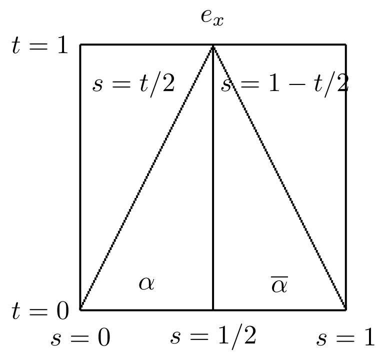


(Work out the corresponding formula for $[e_x][\alpha] = [\alpha] \in \pi_1(X,x)$.)
Proof that $[\alpha][\overline{\alpha}] = [e_x] \in \pi_1(X,x)$
Define a rel $\{0,1\}$ homotopy

$$h \ : \ \alpha \cdot \overline{\alpha} \simeq e_x \ : \ I \to X$$

by

$$h \ : \ I \times I \to X \ ; \ (s,t) \mapsto \begin{cases} x & \text{if } 0 \leqslant s \leqslant t/2 \\ \alpha(2s-t) & \text{if } t/2 \leqslant s \leqslant 1/2 \\ \alpha(2-2s-t) & \text{if } 1/2 \leqslant s \leqslant 1-t/2 \\ x & \text{if } 1-t/2 \leqslant s \leqslant 1 \ . \end{cases}$$



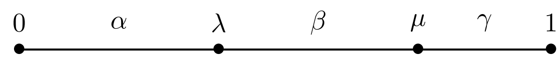
Again, think what happens at each time $t \in I$: the path

$$h_t \;:\; I \to X \;;\; s \mapsto h_t(s) = h(s,t)$$

is constant on $[0, t/2]$, goes along the restriction $\alpha| : [0, 1-t] \to X$ (i.e. using only a part of α) at twice the speed on $[t/2, 1/2]$, then along the restriction $\overline{\alpha}| : [t, 1] \to X$ at twice the speed on $[1/2, 1-t/2]$, and stays constant on $[1-t/2, 1]$. Note that $\alpha(1-t) = \overline{\alpha}(t)$ is essential for continuity. The homotopy h starts at $h_0 = \alpha \cdot \overline{\alpha}$ and ends at $h_1 = e_x$. (Work out the corresponding formula for $[\overline{\alpha}][\alpha] = [e_x]$.)

Proof that $([\alpha][\beta])[\gamma] = [\alpha]([\beta][\gamma]) \in \pi_1(X, x)$ **(associativity of multiplication)**
Let $\alpha, \beta, \gamma : I \to X$ be paths which send each endpoint to $x \in X$. For $0 < \lambda < \mu < 1$ let $c(\lambda, \mu) : I \to X$ be the path defined by

$$c(\lambda,\mu) \;:\; I \to X \;;\; s \mapsto \begin{cases} \alpha(s/\lambda) & \text{if } 0 \leqslant s \leqslant \lambda \\ \beta((s-\lambda)/(\mu-\lambda)) & \text{if } \lambda \leqslant s \leqslant \mu \\ \gamma((s-\mu)/(1-\mu)) & \text{if } \mu \leqslant s \leqslant 1\ . \end{cases}$$

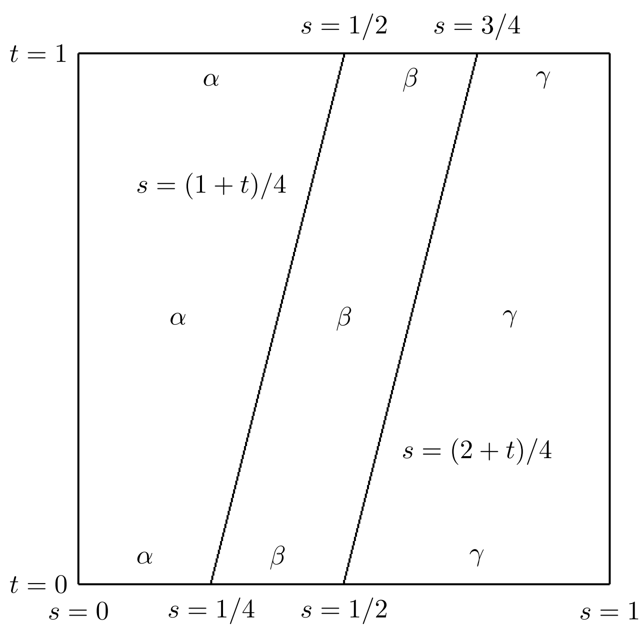


The path starts by going along α at $1/\lambda$ the speed on $[0, \lambda]$, followed by going along β at $1/(\mu - \lambda)$ the speed on $[\lambda, \mu]$, and finish by going along γ at $1/(1-\mu)$ the speed on $[\mu, 1]$. From the definitions

$$([\alpha][\beta])[\gamma] \;=\; c(1/4, 1/2) \;:\; I \to X \;;\; s \mapsto \begin{cases} \alpha(4s) & \text{if } 0 \leqslant s \leqslant 1/4 \\ \beta(4s-1) & \text{if } 1/4 \leqslant s \leqslant 1/2 \\ \gamma(2s-1) & \text{if } 1/2 \leqslant s \leqslant 1 \end{cases}$$

and

$$[\alpha]([\beta][\gamma]) \;=\; c(1/2, 3/4) \;:\; I \to X \;;\; s \mapsto \begin{cases} \alpha(2s) & \text{if } 0 \leqslant s \leqslant 1/2 \\ \beta(4s-2) & \text{if } 1/2 \leqslant s \leqslant 3/4 \\ \gamma(4s-3) & \text{if } 3/4 \leqslant s \leqslant 1\ . \end{cases}$$



Finally, construct a homotopy rel $\{0, 1\}$

$$h \ : \ ([\alpha][\beta])[\gamma] \simeq [\alpha]([\beta][\gamma]) \ : \ I \to X$$

by

$$\begin{aligned} h_t &= c((1-t)/4 + t/2, (1-t)/2 + t(3/4)) \\ &= c((1+t)/4, (2+t)/4) \ : \ I \to X \end{aligned}$$

with $h_0 = c(1/4, 1/2)$, $h_1 = c(1/2, 3/4)$. End of proof of 1.88.

The fundamental group $\pi_1(X, x)$ of a space X at a point $x \in X$ is defined geometrically, in terms of paths $\alpha : I \to X$ such that $\alpha(0) = \alpha(1) = x$, or equivalently in terms of loops $\omega : S^1 \to X$ such that $\omega(1) = x \in X$. A calculation of $\pi_1(X, x)$ is an algebraic description. In general, it is quite difficult to compute $\pi_1(X, x)$, unless there is a geometric reason for it to be the trivial group $\{1\}$.

A space determines a group. A continuous map $f : X \to Y$ induces a group morphism

$$f_* \ : \ \pi_1(X, x) \to \pi_1(Y, f(x)) \ ; \ [\alpha] \mapsto [f\alpha]$$

for any $x \in X$.

Definition 1.89 *Let X be a space, and $x \in X$. A continuous map $f : X \to Y$ is a* homotopy equivalence rel $\{x\}$ *if there exists a continuous map $g : Y \to X$ such that $g(f(x)) = x$, a homotopy rel $\{x\}$ $h : gf \simeq 1_X : X \to X$ (with $h(x, t) = f(x)$ for $t \in I$) and a homotopy rel $\{f(x)\}$ $k : fg \simeq 1_Y : Y \to Y$.*

Proposition 1.90 *(i) If $f, g : X \to Y$ are continuous maps which are related by a rel $\{x\}$ homotopy $h : f \simeq g : X \to Y$ then*

$$f_* \ = \ g_* \ : \ \pi_1(X, x) \to \pi_1(Y, f(x)) \ .$$

(ii) If $f : X \to Y$ is a homotopy equivalence rel $\{x\}$ then f_* is an isomorphism, with inverse

$$(f_*)^{-1} \ = \ g_* \ : \ \pi_1(Y, f(x)) \to \pi_1(X, x) \ .$$

Remark 1 *If $f : X \to Y$ is a homotopy equivalence (not just rel $\{x\}$) then $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$ is an isomorphism.*

1.7 Covering spaces

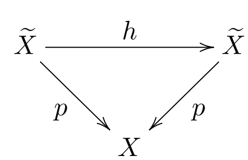
Definition 1.91 *A* covering space *of a space X with* fibre *the discrete space F is a space $\widetilde{X}$ with* covering projection *a continuous map $p : \widetilde{X} \to X$ such that for each $x \in X$ there exists an open subset $U \subseteq X$ with $x \in U$, and with a homeomorphism $\phi_U : F \times U \to p^{-1}(U)$ such that*

$$p \circ \phi_U(a, u) \ = \ u \in U \subseteq X \quad (a \in F, u \in U) \ .$$

In particular, for each $x \in X$ $p^{-1}(x)$ is homeomorphic to F. *[AT56]*

A covering projection $p : \widetilde{X} \to X$ is a 'local homeomorphism': for each $\widetilde{x} \in \widetilde{X}$ there exists an open subset $U \subseteq \widetilde{X}$ such that $\widetilde{x} \in U$ and $U \to p(U); u \mapsto p(u)$ is a homeomorphism, with $p(U) \subseteq X$ an open subset.

Definition 1.92 Given a covering projection $p : \widetilde{X} \to X$ let $\text{Homeo}_p(\widetilde{X})$ be the subgroup of $\text{Homeo}(\widetilde{X})$ consisting of the homeomorphisms $h : \widetilde{X} \to \widetilde{X}$ such that $ph = p : \widetilde{X} \to X$, i.e. such that the diagram



is commutative. Such h are called *covering translations.* □

Definition 1.93 *A covering projection* $p : \widetilde{X} \to X$ *with fibre* F *is* trivial *if there exists a homeomorphism* $\phi : F \times X \to \widetilde{X}$ *such that*

$$p \circ \phi(a, x) \;=\; x \in X \quad (a \in F,\ x \in X)\ .$$

A particular choice of ϕ *is a* trivialisation *of* p.

Example 1.94 *(i) For any space* X *and discrete space* F *the covering projection*

$$p \ :\ \widetilde{X} \;=\; F \times X \to X\ ;\ (a, x) \mapsto x$$

is trivial, with the identity trivialization $\phi = 1 : F \times X \to \widetilde{X}$.
(ii) The continuous map

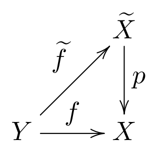
$$p \ :\ \mathbb{R} \to S^1\ ;\ x \mapsto e^{2\pi i x}$$

is a covering projection with fibre $\mathbb{Z}$. *Note that* p *is not trivial, since* $\mathbb{R}$ *is not homeomorphic to* $\mathbb{Z} \times S^1$ *(although [Exercise] there does exist a bijection* $\phi : \mathbb{Z} \times S^1 \cong \mathbb{R}$ *such that* $p \circ \phi : \mathbb{Z} \times S^1 \to S^1$ *is the projection). The group of covering translations is [Exercise] the infinite cyclic group* $\mathbb{Z}$.

Definition 1.95 *Let* $p : \widetilde{X} \to X$ *be a covering projection. A* lift *of a continuous map* $f : Y \to X$ *is a continuous map* $\widetilde{f} : Y \to \widetilde{X}$ *such that*

$$p(\widetilde{f}(y)) \;=\; f(y) \in X \quad (y \in Y)$$

so that there is defined a commutative diagram



Example 1.96 *For the trivial covering projection* $p : \widetilde{X} = F \times X \to X$ *of Example 1.93 define a lift of any continuous map* $f : Y \to X$ *by choosing a point* $a \in F$ *and setting*

$$\widetilde{f}_a \ :\ Y \to \widetilde{X} \;=\; F \times X\ ;\ y \mapsto (a, f(y))\ .$$

If Y *is path-connected every lift of* f *is of this type, and the function* $a \mapsto \widetilde{f}_a$ *defines a one-one correspondence between the points* $a \in F$ *and the lifts* $\widetilde{f}$ *of* f.

Theorem 1.97 (Path lifting property) *Let* $p : \widetilde{X} \to X$ *be a covering projection with fibre* F. *Let* $x_0 \in X$, $\widetilde{x}_0 \in \widetilde{X}$ *be such that* $p(\widetilde{x}_0) = x_0 \in X$.
(i) *Every path* $\alpha : I \to X$ *with* $\alpha(0) = x_0 \in X$ *has a unique lift to a path* $\widetilde{\alpha} : I \to \widetilde{X}$ *such that* $\widetilde{\alpha}(0) = \widetilde{x}_0 \in \widetilde{X}$.
(ii) *Let* $\alpha, \beta : I \to X$ *be paths with* $\alpha(0) = \beta(0) = x_0 \in X$, *and let* $\widetilde{\alpha}, \widetilde{\beta} : I \to \widetilde{X}$ *be the lifts with* $\widetilde{\alpha}(0) = \widetilde{\beta}(0) = \widetilde{x}_0 \in \widetilde{X}$ *given by* (i). *Every rel* $\{0, 1\}$ *homotopy*

$$h \ :\ \alpha \;\simeq\; \beta \ :\ I \to X$$

has a unique lift to a rel $\{0, 1\}$ *homotopy*

$$\widetilde{h} \ :\ \widetilde{\alpha} \;\simeq\; \widetilde{\beta} \ :\ I \to \widetilde{X}$$

and in particular

$$\widetilde{\alpha}(1) \;=\; \widetilde{h}(1, t) \;=\; \widetilde{\beta}(1) \in \widetilde{X} \quad (t \in I)\ . \qquad \text{[AT60]}$$

Definition 1.98 *Given a covering projection* $p : \widetilde{X} \to X$ *and a path* $\alpha : I \to X$ *use the path lifting property (1.97) define the* fibre transport *bijection*

$$\alpha_{\#} \ :\ p^{-1}(\alpha(0)) \to p^{-1}(\alpha(1))\ ;\ \widetilde{x} \mapsto \widetilde{\alpha}_{\widetilde{x}}(1)$$

where $\widetilde{\alpha}_{\widetilde{x}} : I \to \widetilde{X}$ *is the unique lift of* α *with*

$$\widetilde{\alpha}_{\widetilde{x}}(0) \;=\; \widetilde{x} \in \widetilde{X}\ .$$

Proposition 1.99 *A covering projection $p : Y \to X$ of path-connected spaces induces an injective group morphism $p_* : \pi_1(Y) \to \pi_1(X)$.*

Proof If $\omega : S^1 \to Y$ is a loop at $y \in Y$ such that there exists a homotopy $h : p\omega \simeq e_{p(y)} : S^1 \to X$ rel 1, then h can be lifted to a homotopy $\widetilde{h} : \omega \simeq e_y : S^1 \to Y$ rel 1, by the relative version of 1.97. □

Recall that a subgroup $H \subseteq G$ is *normal* if $xH = Hx$ for all $x \in G$, in which case there is defined a quotient group G/H with a canonical surjection $G \to G/H$.

Definition 1.100 *A covering projection $p : Y \to X$ of path-connected spaces is* regular *if $p_*(\pi_1(Y)) \subseteq \pi_1(X)$ is a normal subgroup.*

Here is a very general construction of regular covering projections:

Theorem 1.101 *Given a space Y and a subgroup $G \subseteq \mathrm{Homeo}(Y)$ define an equivalence relation $\sim$ on Y by*

$$y_1 \sim y_2 \text{ if there exists } g \in G \text{ such that } y_2 = g(y_1)$$

and write

$$p \ : \ Y \to X \ = \ Y/\!\sim \ = \ Y/G \ ; \ y \mapsto p(y) \ = \ \text{equivalence class of } y \ .$$

Suppose that for each $y \in Y$ there exists an open subset $U \subseteq Y$ such that $y \in U$ and

$$g(U) \cap U = \emptyset \text{ for } g \neq 1 \in G \ .$$

(Such an action of a group G on a space Y is called free *and* properly discontinuous*). Then $p : Y \to X$ is a covering projection with fibre G. Furthermore, if Y is path-connected then so is X, p is a regular covering and the group of covering translations of p is $\mathrm{Homeo}_p(Y) = G \subset \mathrm{Homeo}(Y)$.* [AT61,72]

Proof The subset $p(U) \subseteq X$ is open, since

$$p^{-1}p(U) \ = \ \{gu \,|\, g \in G, u \in U\} \ = \ \bigcup_{g \in G} g(U) \subseteq Y$$

is open, with an evident homeomorphism

$$\phi_{p(U)} \ : \ G \times p(U) \to p^{-1}p(U) \ ; \ (g, p(u)) \mapsto gu \ .$$

If $h \in \mathrm{Homeo}_p(Y)$ then for any $y \in Y$ there is a unique $g_y \in G$ such that $h(y) = g_y(y) \in Y$ $(y \in Y)$. If Y is path-connected the continuous map $Y \to G; y \mapsto g_y$ is constant (since G is discrete), so $g_y = h \in \mathrm{Homeo}_p(Y) = G$. □

Remark 2 *Every regular covering projection $p : Y \to X$ with X, Y path-connected arises as in Theorem 1.101 from a free action of a group $G = \pi_1(X)/p_*(\pi_1(Y))$ on Y, or equivalently from a surjection $\pi_1(X) \to G$.*

Theorem 1.102 *For a regular covering projection $p : Y \to X$ of a path-connected space X there is defined an isomorphism of groups*

$$\pi_1(X)/p_*(\pi_1(Y)) \ \cong \ \mathrm{Homeo}_p(Y) \ .$$

Proof Let $x_0 \in X$, $y_0 \in Y$ be base points such that $p(y_0) = x_0$. Every closed path $\alpha : I \to X$ with $\alpha(0) = \alpha(1) = x_0$ has a unique lift to a path $\widetilde{\alpha} : I \to Y$ such that $\widetilde{\alpha}(0) = y_0$. The function

$$\pi_1(X, x_0)/p_*\pi_1(Y, y_0) \to p^{-1}(x_0) \ ; \ \alpha \mapsto \widetilde{\alpha}(1)$$

is a bijection. For each $y \in p^{-1}(x_0)$ there is a unique covering translation $h_y \in \mathrm{Homeo}_p(Y)$ such that

$$h_y(y_0) \ = \ y \in Y \ .$$

The function

$$p^{-1}(x_0) \to \mathrm{Homeo}_p(Y) \ ; \ y \mapsto h_y$$

is a bijection, with inverse $h \mapsto h(\widetilde{x}_0)$. The composite bijection

$$\pi_1(X, x_0)/p_*(\pi_1(Y)) \to p^{-1}(x_0) \to \mathrm{Homeo}_p(Y)$$

is an isomorphism of groups. □

Example 1.103 *For each $n \in \mathbb{Z}$ the translation of $\mathbb{R}$ by n units to the right defines a homeomorphism*

$$h_n \ : \ \mathbb{R} \to \mathbb{R} \ ; \ x \mapsto x + n$$

with $h_n h_m = h_{m+n}$. The infinite cyclic subgroup

$$G \ = \ \{h_n \,|\, n \in \mathbb{Z}\} \subset \mathrm{Homeo}(\mathbb{R})$$

satisfies the hypothesis of Theorem 1.101, so that

$$p \ : \ \mathbb{R} \to \mathbb{R}/G \ = \ \mathbb{R}/\mathbb{Z} \ = \ S^1 \ ; \ x \mapsto e^{2\pi i x}$$

is a regular covering projection with fibre $G = \mathbb{Z}$ and by Theorem 1.102

$$\pi_1(S^1) \ = \ \mathrm{Homeo}_p(\mathbb{R}) \ = \ G \ = \ \mathbb{Z} \subset \mathrm{Homeo}(\mathbb{R}) \ .$$

Every loop $\omega : S^1 \to S^1$ can be lifted to a path $\alpha : I \to \mathbb{R}$ such that

$$\omega(e^{2\pi i t}) \ = \ e^{2\pi i \alpha(t)} \in S^1 \ \ (t \in I) \ .$$

The degree of ω is defined by

$$\mathrm{degree}(\omega) \ = \ \alpha(1) - \alpha(0) \in \mathbb{Z} \ ,$$

and the function

$$\pi_1(S^1) \to \mathbb{Z} \ ; \ \omega \mapsto \mathrm{degree}(\omega)$$

is an isomorphism of groups. A loop ω with degree$(\omega) = d$ *is homotopic to the standard loop with degree d*

$$\omega_d \ : \ S^1 \to S^1 \ ; \ z \mapsto z^d$$

with lift $\alpha_d : I \to \mathbb{R}; t \mapsto dt$.

Definition 1.104 *A space X is* simply-connected *if it is path-connected and $\pi_1(X) = \{1\}$.*

Proposition 1.105 *Every covering projection $p : \widetilde{X} \to X$ of a simply-connected space X is trivial.*

Proof Let F be the fibre. Choose a base point $x_0 \in X$, and an open neighbourhood $U_0 \subseteq X$ of x_0 with a trivialisation

$$\phi_0 \ : \ F \times U_0 \to p^{-1}(U_0)$$

of $p| : p^{-1}(U_0) \to U_0$, i.e. a homeomorphism such that

$$p \circ \phi_0(a, u) \ = \ u \in X \ \ (a \in F, u \in U_0) \ .$$

In particular, there is defined a bijection

$$F \to p^{-1}(x_0) \ ; \ a \mapsto \phi_0(a, x_0) \ .$$

For each $x \in X$ choose a path $\alpha_x : I \to X$ from $\alpha_x(0) = x_0$ to $\alpha_x(1) = x$, and use fibre transport (1.98) to define a homeomorphism

$$\phi \ : \ F \times X \to \widetilde{X} \ ; \ (a, x) \mapsto (\alpha_x)_\#(\phi_0(a, x_0)) \ .$$

The condition $\pi_1(X) = \{1\}$ is needed to prove that ϕ is independent of the choices of paths α_x. □

Example 1.106 *(i) Every covering $p : \widetilde{I} \to I$ is trivial, with a homeomorphism $\phi : F \times I \to \widetilde{I}$ such that $p\phi(a, x) = x$.*
(ii) For any discrete space F and bijection $\sigma : F \to F$ define a covering $p : \widetilde{S}^1 \to S^1$ with fibre F by

$$p \ : \ \widetilde{S}^1 \ = \ F \times I/\{(x, 0) \sim (\sigma(x), 1)\} \to S^1 = I/\{0 \sim 1\} \ ; \ [x, t] \mapsto [t] \ .$$

In fact, every covering $p : \widetilde{S}^1 \to S^1$ arises in this way: define the closed path

$$\alpha \ : \ I \to S^1 \ ; \ t \mapsto e^{2\pi i t}$$

with $\alpha(0) = \alpha(1) = 1 \in S^1$, and note that the fibre transport is a bijection

$$\alpha_\# \ : \ F \ = \ p^{-1}(1) \to F \ = \ p^{-1}(1)$$

such that

$$p \ : \ \widetilde{S}^1 \ = \ F \times I/\{(x, 0) \sim (\alpha_\#(x), 1)\} \to S^1 = I/\{0 \sim 1\} \ ; \ [x, t] \mapsto [t] \ .$$

Definition 1.107 *A covering projection* $p : \widetilde{X} \to X$ *of a path-connected space* X *is* universal *if* $\widetilde{X}$ *is simply-connected.*

Example 1.108 *The covering projection* $p : \mathbb{R} \to S^1$ *is universal.*

A space X is *locally path connected* if for each $x \in X$ and for each open subset $U \subseteq X$ with $x \in U$ there is a path-connected open subset $V \subseteq U$ with $x \in V$. (Main example: open subsets of $\mathbb{R}^n$).

Theorem 1.109 *Let* X *be a path-connected locally path-connected space with a universal covering projection* $p : \widetilde{X} \to X$*. Let* $x_0 \in X$*,* $\widetilde{x}_0 \in \widetilde{X}$ *be base points such that* $p(\widetilde{x}_0) = x_0$*.*
(i) *The function*

$$\pi_1(X, x_0) \to p^{-1}(x_0) \ ; \ \alpha \mapsto \alpha_\#(\widetilde{x}_0)$$

is a bijection.
(ii) *For each* $y \in p^{-1}(x_0)$ *there is a unique covering translation* $h_y \in \mathrm{Homeo}_p(\widetilde{X})$ *such that*

$$h_y(\widetilde{x}_0) \ = \ y \in \widetilde{X} \ .$$

The function

$$p^{-1}(x_0) \to \mathrm{Homeo}_p(\widetilde{X}) \ ; \ y \mapsto h_y$$

is a bijection, with inverse $h \mapsto h(\widetilde{x}_0)$*. The composite bijection*

$$\pi_1(X, x_0) \to p^{-1}(x_0) \to \mathrm{Homeo}_p(\widetilde{X})$$

is an isomorphism of groups. [AT61]

Remark 3 *If* $p : \widetilde{X} \to X$ *is a universal covering projection satisfying the hypothesis of Theorem 1.109 then for any subgroup*

$$G \subseteq \pi_1(X) \ = \ \mathrm{Homeo}_p(\widetilde{X})$$

there is defined a universal covering projection

$$q \ : \ \widetilde{Y} \ = \ \widetilde{X} \to Y \ = \ \widetilde{X}/G$$

also satisfying the hypothesis of 1.109, with

$$\pi_1(Y) \ = \ \mathrm{Homeo}_q(\widetilde{Y}) \ = \ G \ .$$

If $G \subseteq \pi_1(X)$ *is a normal subgroup the projection* $r : Y \to X$ *is a covering projection with*

$$\mathrm{Homeo}_r(Y) \ = \ \pi_1(X)/G \ .$$

Remark 4 *Theorem 1.109 gives a geometric method for computing the fundamental group of a path-connected space* X *which admits a universal covering* $p : \widetilde{X} \to X$*, namely*

$$\pi_1(X, x_0) \ = \ \mathrm{Homeo}_p(\widetilde{X}) \ = \ p^{-1}(x_0) \ .$$

For any path-connected space X *and* $x_0 \in X$ *let* $\widetilde{X}$ *be the topological space of equivalence class of paths* $\alpha : I \to X$ *such that* $\alpha(0) = x_0$*, with* $\alpha \sim \alpha'$ *if there exists a rel* $\{0, 1\}$ *homotopy* $\beta : \alpha \simeq \alpha' : I \to X$*, and*

$$p \ : \ \widetilde{X} \to X \ ; \ \alpha \mapsto \alpha(1) \ .$$

It is a theorem that p *is the universal covering projection of* X *with fibre* $F = p^{-1}(x_0) = \pi_1(X, x_0)$ *if* X *is* semi-locally simply-connected*, meaning that for every* $x \in X$ *there exists an open subset* $U \subseteq X$ *with* $x \in U$ *such that the inclusion* $i : U \to X$ *induces the trivial homomorphism* $i_* = 1 : \pi_1(U, x) \to \pi_1(X, x)$ *(in which case* $p^{-1}(U)$ *is homeomorphic to* $U \times \pi_1(X, x)$*). In general, this is too synthetic a construction of the universal cover to be of use in the computation of* $\pi_1(X)$*. In practice, a geometrically interesting space* X *has a geometrically interesting universal cover* $\widetilde{X}$*, and this can be used to compute* $\pi_1(X)$*. For example, a smooth atlas* $\mathcal{A}$ *on an* m*-dimensional manifold* M *can be used to construct a universal cover* $\widetilde{M}$*, which is again an* m*-dimensional manifold with a smooth atlas* $\widetilde{\mathcal{A}}$*.*

1.8 The higher homotopy groups $\pi_*(X)$

The higher homotopy group $\pi_n(X,x)$ is defined for $n \geqslant 1$ to be the set of based homotopy classes of continuous maps $\omega : S^n \to X$ such that $\omega(1) = x \in X$, where $1 = (1,0,\dots,0) \in S^n$. In order to define the group law it is convenient to identify S^n with the quotient space $I^n/\partial I^n$, with $I^n = I \times \cdots \times I$ the unit n-cube and ∂I^n its boundary. There is an evident one-one correspondence between the continuous maps $\alpha : I^n \to X$ such that $\alpha(\partial I^n) = \{x\}$ and the continuous maps $\omega : S^n \to X$ such that $\omega(1) = x \in X$. Similarly for homotopies.

Definition 1.110 *The* nth homotopy group $\pi_n(X,x)$ *is the set of rel* ∂I^n *homotopy classes of continuous maps* $\alpha : I^n \to X$ *such that* $\alpha(\partial I^n) = \{x\}$*, with the group law*

$$\pi_n(X,x) \times \pi_n(X,x) \to \pi_n(X,x) \ ; \ ([\alpha],[\beta]) \mapsto [\alpha][\beta] = [\alpha \cdot \beta]$$

given by

$$\alpha \cdot \beta \ : \ I^n \to X \ ; \ (t_1,t_2,\dots,t_n) \mapsto \begin{cases} \alpha(2t_1,t_2,\dots,t_n) & \textit{if } 0 \leqslant t_1 \leqslant 1/2 \\ \beta(2t_1-1,t_2,\dots,t_n) & \textit{if } 1/2 \leqslant t_1 \leqslant 1 \end{cases}$$

and inverses by

$$\pi_n(X,x) \to \pi_n(X,x) \ ; \ [\alpha] \mapsto [-\alpha : (t_1,t_2,\dots,t_n) \mapsto \alpha(1-t_1,t_2,\dots,t_n)] \ .$$

In particular, for $n = 1$ this is just the fundamental group $\pi_1(X,x)$.
The basic properties of the higher homotopy groups have already been stated in Section 1.5.2.