The topology of manifolds
Budapest August 2011
Solutions to problems

1 Monday

1. (DC) (a) Let $H = \mathbb{C} \to H \to S^2$ denote the Hopf bundle which we think of to begin as a complex line bundle over S^2. The fibre-wise projectivisation of H is a smooth bundle over S^2 with fibre $\mathbb{C}P^1$: denote it by $P(H)$. Let $H_{\mathbb{R}} \to S^2$ denote the real 2-plan bundle defined by H. Checking the definitions, we see that $P(H)$ is precisely $S(H_{\mathbb{R}} \oplus \underline{\mathbb{R}})$, the sphere bundle of the $\mathbb{R}^3$-bundle obtained by stabilising $H_{\mathbb{R}}$ with a trivial real line bundle $\underline{\mathbb{R}}$. Thus we have a diffeomorphism

$$P(H) \cong S(H_{\mathbb{R}} \oplus \underline{\mathbb{R}}).$$

Since the clutching function for the Hopf bundle generates $\pi_1(SO(2)) \cong \mathbb{Z}$ and the stabilisation map $\pi_1(SO(2)) \to \pi_1(SO(3)) \cong \mathbb{Z}/2$ is onto, we see see that $H_{\mathbb{R}} \oplus \underline{\mathbb{R}}$ is the non-trivial $\mathbb{R}^3$-bundle over S^2 and so $S(H_{\mathbb{R}} \oplus \underline{\mathbb{R}}) = S^2 \tilde{\times} S^2$ by definition.

On the other hand, let us return to $P(H)$, the fibrewise projectivisation of H: the fibres of $P(H)$ are the space of complex lines in the complex vector space $H_x \oplus \mathbb{C}$ where $H_x \subset H$, $x \in S^2$. Up to homeomoprhism each fibre $P(H)_x$ isf ormed by adding a line a infinity to $H_x \cong \mathbb{C}$. Thus we see that $P(H) \to S^2$ has a pair of canonical sections $s_0 \colon S^2 \to P(H)$ and $s_\infty \colon S^2 \to P(H)$, respectively the zero section and the section at infinity. Clearly a tubular neighbourhood of $s_0(S^2)$, N_0, is diffeomorphic to H, whereas a tubular neighbourhood of $s_\infty(S^2)$, N_∞, will be diffeomorphic to $\bar{H}$, the complex conjugate bundle of H. It is not hard to see that the manifold $P(H) - N_0 - N_\infty$ is diffeomorphic to $S(H) \times \mathbb{R}$ where $S(H) \cong S^2$ is the unit sphere bundle of H. Finally, it is well known that $\mathbb{C}P^2 \cong H \cup_{S^3} D^4$ and that $-\mathbb{C}P^2 \cong \bar{H} \cup_{S^3} D^4$. Putting all of this together we conclude that there is a diffeomorphism

$$P(H) \cong \mathbb{C}P^2 \sharp (-\mathbb{C}P^2)$$

and from the diffeomorphisms of the first paragraph we conclude that $\mathbb{C}P^2 \sharp (-\mathbb{C}P^2) \cong S^2 \tilde{\times} S^2$.

(b) In part (a) our arguments showed that the bundle $S^2 \tilde{\times} S^2 \to S^2$ has a section. It follows that the long exact homotopy sequence for this fibration splits and that there are isomorphisms

$$\pi_i(S^2 \tilde{\times} S^2) \cong \pi_i(S^2) \times \pi_i(S^2) \cong \pi_i(S^2 \times S^2).$$

We can show that $S^2 \tilde{\times} S^2$ and $S^2 \times S^2$ are not homotopy equivalent by looking at their intersections forms. Using part (a) we see that the intersection form of $S^2 \tilde{\times} S^2$ is represented by the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and in particular is *odd*. On the other hand, the intersection form of $S^2 \times S^2$ is represented by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and in particular is *even*. Since the parity of a nonsingular symmetric bilinear form over the integers is an isometry invariant and since the intersection form of an oriented manifold is an oriented homotopy invariant we conclude that there is no orientation preserving homotopy equivalence from $S^2 \times S^2$ to $S^2 \tilde{\times} S^2$. But of course the parity of form (H, ϕ) is the same as the parity of the form $(H, -\phi)$ and so we can strengthen our conclusion to deny the existence of any homotopy equivalence.

2. (Béla Rácz) (a) Indeed, $1 - t - t^4 \in \mathbb{Z}[\mathbb{Z}/5]$ is a unit:

$$(1-t-t^4)(1-t^2-t^3) = \big((1-t-t^4) + (-t^2+t^3+t) + (-t^3+t^4+t^2)\big) = 1.$$

(b) Let $\omega = e^{\frac{2\pi i}{5}}$. Since $\mathbb{Z}[\mathbb{Z}/5]$ is cyclic with generator t, and $\omega^5 = 1$, there is a unique homomorphism to the multiplicative group of the complex numbers, $\varphi : \mathbb{Z}/5 \to \mathbb{C}^\times$, such that $\varphi(t) = \omega$. Its extension $\phi : \mathbb{Z}[\mathbb{Z}/5] \to \mathbb{C}$ where $\phi(g) = \varphi(g)$ for $g \in \mathbb{Z}/5$ is a unital ring homomorphism.

However, the image of the unit identified in (a) is:

$$\phi(1-t-t^4) = 1 - \omega - \omega^4 = 1 - 2\cos\left(\frac{2\pi}{5}\right) \in (0,1) \subset \mathbb{C} \setminus S^1.$$

(c) The homomorphism ϕ induces a homomorphism $\bar{\phi} : GL(\mathbb{Z}[\mathbb{Z}/5]) \to GL(\mathbb{C})$. Let us consider the following composition of group homomorphisms:

$$GL(\mathbb{Z}[\mathbb{Z}/5]) \xrightarrow{\bar{\phi}} GL(\mathbb{C}) \xrightarrow{\det} \mathbb{C}^\times \xrightarrow{|\cdot|} (\mathbb{R}_+)^\times$$

So if $\psi(A) = |\det(\bar{\phi}(A))|$, then $\psi : GL(\mathbb{Z}[\mathbb{Z}/5]) \to (\mathbb{R}_+)^\times$ is a group homomorphism.

If $H = (GL(\mathbb{Z}[\mathbb{Z}/5]))'$ is the commutator subgroup, then $\det(\bar{\phi}(H)) = 1$, since we have the identity $H = \langle [A,B] | A, B \in GL(\mathbb{Z}[\mathbb{Z}/5]) \rangle$, and $\det([\bar{\phi}(A), \bar{\phi}(B)]) = 1$. Thus, the map $\psi : K_1(\mathbb{Z}/5) = GL(\mathbb{Z}[\mathbb{Z}/5])/H \to (\mathbb{R}_+)^\times$ is well-defined.

We can obtain $\mathrm{Wh}(\mathbb{Z}/5)$ as $K_1(\mathbb{Z}/5)/K$, where the kernel K is generated by the image of group elements (as 1×1 matrices) in the matrix group: $K = (\{\pm g | g \in \mathbb{Z}/5\} H)/H$.

We note that $\psi(K) = 1$, since $|\varphi(g)| = 1, \forall g \in \mathbb{Z}/5$. Therefore, $\psi : \mathrm{Wh}(\mathbb{Z}/5) = \mathrm{K}_1(\mathbb{Z}/5)/\mathrm{K} \to (\mathbb{R}_+)^\times$ is still well-defined.

Now let us consider the unit $u = 1 - t - t^4 \in \mathbb{Z}[\mathbb{Z}/5]$. Since u is a unit, it embeds to $GL(\mathbb{Z}[\mathbb{Z}/5])$ as a 1×1 invertible matrix. Let u' be the image of u in $\mathrm{Wh}(\mathbb{Z}/5)$, after factoring by H and K. The image of u', namely $\psi(u') = 1 - 2\cos\left(\frac{2\pi}{5}\right) < 1$ has an infinite order in the group $(\mathbb{R}_+)^\times$. Thus, $\psi(\mathrm{Wh}(\mathbb{Z}/5))$ contains an infinite cyclic group, so $\mathrm{Wh}(\mathbb{Z}/5)$ must also be infinite.

3. (Andrzej Czarnecki) Assume that $BSO \to BO$ is the normal 0-type of M. Homotopy groups aside, this means that the normal bundle has a lift to BSO and as such, it is orientable. The orientability of M follows. Assume that M is orientable, so this gives a lift of the normal map to BSO. This lift is a 1-equivalence, as $\pi_1(BSO) = 0$. As the fibration is actually a twofold covering, the homotopy groups of the fiber are trivial in required range and the argument is complete.

   Assume that $B\mathrm{Spin} \times K(\pi, 1) \to BSO$ is a normal 1-type of M. Again, this gives a lift of the normal bundle to $B\mathrm{Spin}$ and thus M is spinable. In the other direction, assume that M is spinable. Then we have a lifting of the normal bundle to a map $\bar{\nu} \colon M \to B\mathrm{Spin}$. To find a space 2-equivalent to M, we take the Cartesian product $B\mathrm{Spin} \times K(\pi, 1)$ and enhance the lift $\bar{\nu}$ by taking the product with a map $u : M \to K(\pi, 1)$ inducing an isomorphism of fundamental groups (either construct this by hand and obstruction theory or use the map classifying the universal covering). The homotopy fiber has trivial groups in the required range (the last part is a consequence of Spin being the universal covering of O and the functor B shifting the homotopy groups; in particular, $\pi_2(B\mathrm{Spin} \times K(\pi, 1)) = 0$).

4. (Johannes Nordstrom) Let M by stably k-parallelisable, and let $M^{(k)}$ be a k-skeleton. Clearly there is a lift $\bar{\nu} : M^{(k)} \to BO\langle j{+}2\rangle$ of the restriction of the stable normal map of M to the k-skeleton. This can be extended to $\bar{\nu} : M \to BO\langle j{+}2\rangle$ provided that $j \leq k - 1$, since then $\pi_i BO\langle j{+}2\rangle \cong \pi_i BO$ for all $i > k$. The product of $\bar{\nu}$ with the $(j + 1)$-equivalence $M \to P_j(M)$ then gives a normal j-smoothing

$$M \to P_j(M) \times BO\langle j{+}2\rangle.$$

   The space $BO\langle j{+}2\rangle$ is j-coconnected, so the obvious fibration $P_j(M) \times BO\langle j{+}2\rangle \to BO$ is the normal j-type of M. For special cases, see Monday (3) above:

   - M is orientable $\Leftrightarrow$ 1-parallelisable $\Rightarrow$ the normal 0-type is $BO\langle 2\rangle = BSO \to BO$
   - M is spin $\Leftrightarrow$ stably 2-parallelisable $\Rightarrow$ the normal 1-type is

$$P_1(M) \times BO\langle 3\rangle = K(\pi_1 M, 1) \times B\mathrm{Spin} \to BO.$$

5. (Wojciech Politarczyk and DC)

   Let M be $(n - 1)$-connected $2n$-manifold, where $n \geq 3$. We know that $M \setminus D^{2n}$ is a handlebody built with only 0- and n-handles. Thus $N = M \setminus D^{2n}$ is homotopy equivalent to a bouquet of spheres

$$N \simeq \bigvee_{i=1}^{r} S^n,$$

where $r = \operatorname{rank} H_n(M)$. Now the CW-decomposition of the pair (M, N) consists of only one cell D^{2n}, so the pair (M, N) is $(2n-1)$-connected.

Looking at the CW decomposition of the pair $(\prod_{i=1}^r S^n, \bigvee_{i=1}^r S^n)$ we see that it consists of cells of dimension $\geq 2n$. Thus the pair is $(2n-1)$-connected. From the homotopy exact sequence of the pair $(\prod_{i=1}^r S^n, \bigvee_{i=1}^r S^n)$ we see that

$$\pi_{n+1}(\bigvee_{i=1}^{r} S^n) \cong \pi_{n+1}(\prod_{i=1}^{r} S^n) \cong (\mathbb{Z}/2)^r.$$

From the previous consideration we deduce that the following embedding induces an isomorphism on π_{n+1}

$$\bigvee_{i=1}^{r} S^n \hookrightarrow M$$

Thus in order to determine the homomorphism

$$\nu_*\colon \pi_{n+1}(M) \longrightarrow \pi_{n+1}(BO)$$

we need to consider only the following maps

$$S^{n+1} \xrightarrow{\eta_n} S^n \hookrightarrow \bigvee_{i=1}^{r} S^n \longrightarrow M \xrightarrow{\nu} BO.$$

The first thin we notice is that

$$\operatorname{Im}(\nu_*\colon \pi_{n+1}(M) \longrightarrow \pi_{n+1}(BO)) \subset \operatorname{Im}(\eta_n^*\colon KO(S^n) \longrightarrow KO(S^{n+1})).$$

Howver $\eta_n^* = 0$ when $n \equiv 2,3,4,5,6,7 \pmod 8$. When $n \equiv 0,1 \pmod 8$ then $\pi_{n+1}(BO) = \mathbb{Z}/2$. Thus ν_* is either zero or is an epimorphism.

Thus when $n \equiv 2,3,4,5,6,7 \pmod 8$ then the normal n-type of M is given by the following fibration

$$p_{n+2}\colon BO\langle n+2\rangle \longrightarrow BO$$

When $n \equiv 0,1 \pmod 8$ then we have two cases:

(a) if $\nu_*\colon \pi_{n+1}(M) \to \pi_{n+1}(BO)$ is zero then the normal n-type is given by the following fibration

$$p_{n+2}\colon BO\langle n+2\rangle \longrightarrow BO.$$

(b) if the above homomorphism is nonzero, then the normal n-type is given by the following fibration

$$p_{n+1}\colon BO\langle n+1\rangle \longrightarrow BO,$$

notice that $\pi_{n+1}(BO) = \mathbb{Z}/2$, so ν_* is an epimorphism.

The normal k-type of a homotopy sphere.
Suppose that Σ^n is a n-dimensional homotopy sphere. Clearly Σ^n is k-parallelisable for any $k \leq n-1$. Now by Monday (4) we have that the normal k-type of a homotopy n-sphere ($k \leq n-2$) is represented by fibration

$$p_{k+2}\colon BO\langle k{+}2\rangle \longrightarrow BO.$$

Here we used the fact that a homotopy n-sphere has a trivial kth Postnikov stage for $k \leq n-1$: $P_k(\Sigma^n) \simeq \mathrm{pt}$. Recall that $BO\langle j\rangle \to BO$ is the $(j-1)$-connective cover of BO: i.e. $\pi_i(BO\langle j\rangle) = 0$ for $i < j$ and $p_j\colon \pi_i(BO\langle j\rangle) \cong \pi_i(BO)$ for $i \geq j$.

Note that [K, Proposition 1] gives the normal k-types of a homotopy n-sphere but that the notation of [K, §1] is non-standard: the space we denote $BO\langle k{+}2\rangle$ is written $BO\langle k{+}1\rangle$ in [K, §1].

Note also that Kervaire and Milnor [K-M, Theorem 3.1] proved that each homotopy sphere is stably paralellizable. From this we can conclude that the normal $(n-1)$-type of a homotopy n-sphere is

$$p_{n+1}\colon BO\langle n+1\rangle \to BO.$$

The fact that homotopy spheres are stably parallelisable is a fundamental and non-trivial result of differential topology: an elementary or direct proof of this fact would be very interesting. Note that an advantage of the modified surgery in [K] is that in this theory one can start to classify homotopy n-spheres without first proving that they are stably parallelisable.

The normal k-type of a real projective space.
The stable normal bundle of $\mathbb{R}P^n$ in equivalent to $-(n+1)\gamma^1_n$, where γ^1_n denotes the tautological bundle over $\mathbb{R}P^n$. Thus consider a map $q_n\colon \mathbb{R}P^\infty \to BO$ which classifies $-(n+1)\gamma^1_\infty$, where γ^1_∞ is the tautological bundle. The normal k-type $(1 \leq k \leq n-1)$ is represented by the following fibration

$$\mathbb{R}P^\infty \times BO\langle k{+}2\rangle \xrightarrow{q_n \times p_{k+2}} BO \times BO \xrightarrow{\oplus} BO.$$

The normal k-type of a complex projective space.
The case of a complex projective space is completely analogous to the real case. In consequence the normal k-type of $\mathbb{C}P^n$ $(2 \leq k \leq 2n-1)$ is represented by the following fibration

$$\mathbb{C}P^\infty \times BO\langle k{+}2\rangle \xrightarrow{r_n \times p_{k+2}} BO \times BO \xrightarrow{\oplus} BO,$$

where r_n classifies $-(n+1)\bar{\eta}$ and where $\bar{\eta}$ is the dual of the canonical line bundle over $\mathbb{C}P^\infty$.

The normal k-type of an aspherical manifold.
Let M be an aspherical n-manifold. Then the normal k-type of M $(1 \leq k \leq n-1)$ is given by the following fibration

$$M \times BO\langle k{+}2\rangle \xrightarrow{\nu \times p_{k+2}} BO \times BO \xrightarrow{\oplus} BO.$$

The normal k-type of an $(n-1)$-connected $2n$-manifold.
Suppose M is a $(n-1)$-connected $2n$-manifold. Then if $k < n-1$ the normal k-type of M is given by the following fibration

$$p_{k+2}\colon BO\langle k{+}2\rangle \longrightarrow BO.$$

For the case $k = n-1$ recall that the primary obstruction to the trivialising the stable tangent bundle of M defines a homomorphism $\alpha\colon \pi_n(M) \to \pi_n(BO)$ [W, Diffeomorphism invariants of N]. In this case the normal $(n-1)$-type of M is determined by the image of α as follows:

(a) If $\alpha = 0$ then $B^{n-1}(M) = (p_{n+1}\colon BO\langle n{+}1\rangle \to BO)$.
(b) If α is onto then $B^{n-1}(M) = (p_n\colon BO\langle n\rangle \to BO)$.
(c) If $n = 4k$ and $\mathrm{Im}(\alpha) = a \cdot \pi_{4k}(BO) \cong a \cdot \mathbb{Z}$ then

$$B^{n-1}(M) = (BO\langle n\rangle_a \to BO)$$

is the following fibration. The space $BO\langle n\rangle_a$ is the total space of principal $K(\mathbb{Z}/a, n-1)$ bundle over $BO\langle n\rangle$

$$K(\mathbb{Z}/a, n-1) \to BO\langle n\rangle_a \to BO\langle n\rangle$$

classified by any map $BO\langle n\rangle \to K(\mathbb{Z}/a, n)$ which induces a surjection on π_n. Note that here we regard $K(\mathbb{Z}/a, n-1)$ as an abelian topological group with classifying space $K(\mathbb{Z}/a, n)$. The map defining the fibration is the obvious composition $BO\langle n\rangle_a \to BO\langle n\rangle \stackrel{p_n}{\to} BO$.

6. No solution.

7. (Marek Kaluba)
(c) Case $\pi - \mathbb{Z}/2$: We compute $\Omega_4^{Spin} \cong \Omega_4^{Spin}(K(\mathbb{Z}/2,1)) \cong \mathbb{Z}$
To compute will make use of the Atiyah-Hirzebruch Spectral Sequence for Ω_*^{Spin} (generalized) cohomology theory.

$$E^2_{p,q} = H_p(X, \Omega_q^{Spin}(\mathrm{pt})) \Longrightarrow \Omega_{p+q}^{Spin}(X).$$

We will also make use of the following two (not so basic) facts concerning differentials on the second page:

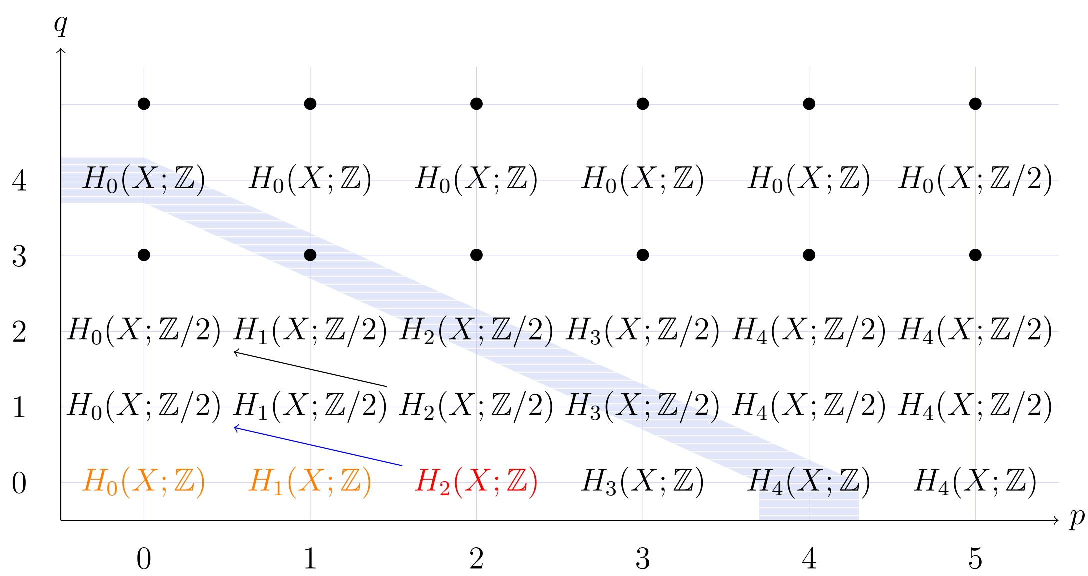
- the differential
$$d_2\colon H_p(X, \Omega_1^{Spin}) \to H_{p-2}(X, \Omega_2^{Spin})$$
connecting the 1-st and the 2-nd row is the **dual** of second Steenrod cohomology operation Sq^2.
- the differential
$$d_2\colon H_p(X, \Omega_0^{Spin}) \to H_{p-2}(X, \Omega_1^{Spin})$$
connecting the 0-th and the 1-st row is the **dual** of second Steenrod cohomology operation Sq^2 composed with reduction to $\mathbb{Z}/2$-coefficients.

Our aim is to compute the fourth diagonal and based on it solve (potentially non-trivial) extension problem. However, we will compute all differentials connecting the first three rows (which is not harder than computing just the needed ones).

Recall low dimensional spin bordism groups (our spectral sequence coefficients) are as follows:

n	0	1	2	3	4	5	6	7	8	...
Ω_n^{Spin}	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}\oplus\mathbb{Z}$	...

From all what we have just said the second page $E^2_{p,q} = H_p(X;\Omega_q^{Spin}(\mathrm{pt}))$ page looks as follows (we use bullet to denote the trivial group):



Let us begin with some general statements based on the diagram:

(a) 4-th row on E^2 remains unchanged in E^3 since all differentials coming to, and leaving are 0.

(b) the same is true for $E^2_{0,0}$, $E^2_{1,0}$.

(c) all differentials coming to 0-th column are zero (this is a general fact about all pages and all spectral sequences), and therefore $E^3_{0,*} = E^2_{0,*}$.

By point 3), the differential coming **from** $E^2_{2,0}$ is 0 and since the differential coming **to** $E^2_{2,0}$ is obviously 0 too, it follows that $E^3_{2,0} = E^2_{2,0}$.

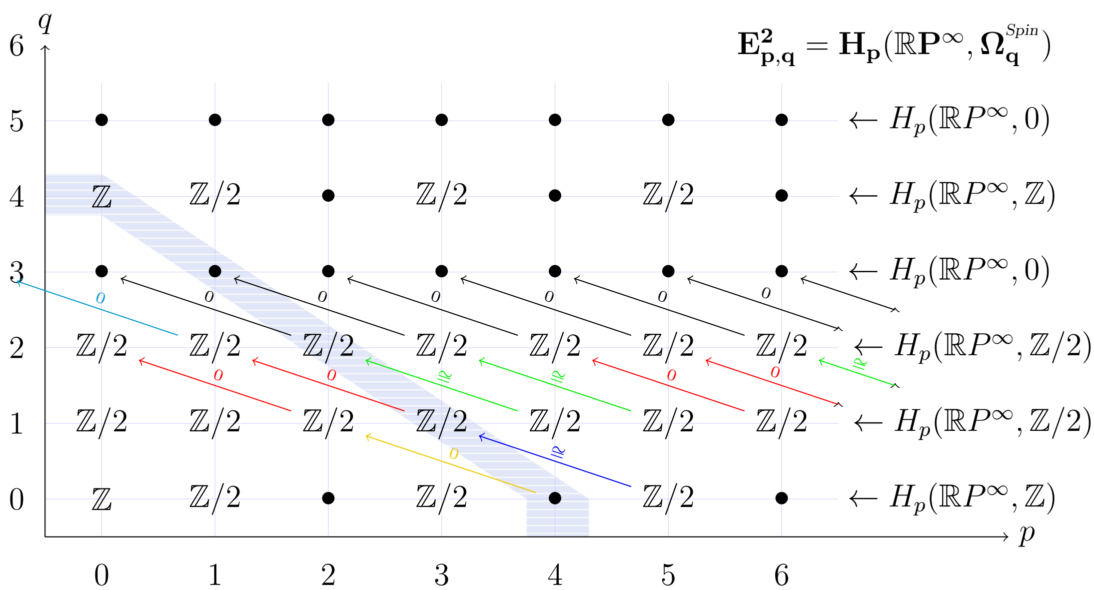
Let us begin with short revision of homology of $K(\mathbb{Z}/2,1)$ with coefficients in $\mathbb{Z}$ and $\mathbb{Z}/2$ (coefficients occuring in our spectral sequence):

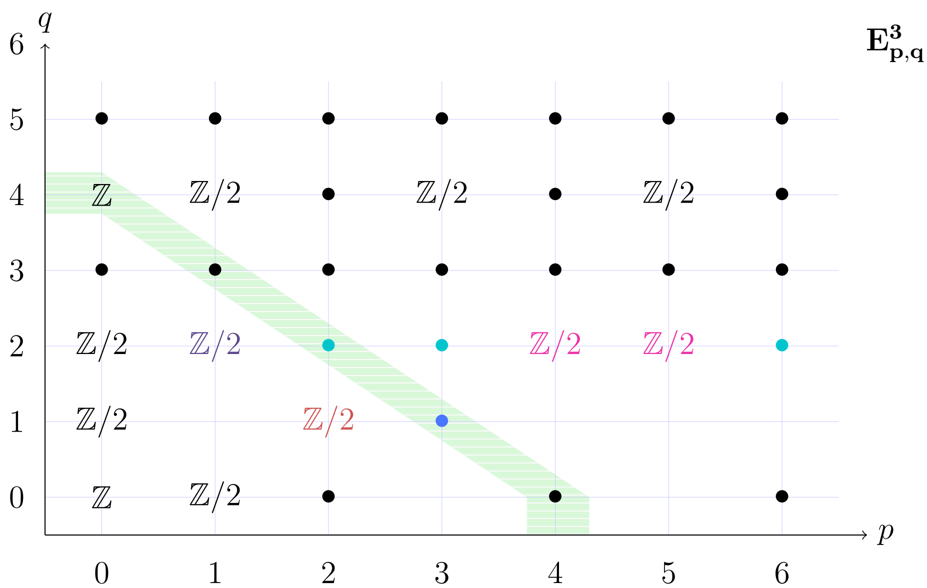
$$H_p(K(\mathbb{Z}/2,1);\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } p = 0, \\ \mathbb{Z}/2 & \text{for } p \text{ odd}, \\ 0 & \text{for } p > 0, p \text{ even} \end{cases}$$

$$H_p(K(\mathbb{Z}/2,1);\mathbb{Z}/2) = \quad \mathbb{Z}/2 \qquad \text{for all } p$$

These two equalities follow easily after identification $K(\mathbb{Z}/2,1) \cong \mathbb{R}P^\infty$ and using the standard CW-structure for the latter.

Substituting this to the diagram above gives us:





We have already set some terms on the E^3 page and these are as pictured in the following diagram (in black).

To compute the remaining part of the second page we need to recall some basic properties of Steenrod operations (see e.g. Milnor, Stasheff, *Characteristic Classes*).

(a) $Sq^n \colon H^k(X;\mathbb{Z}/2) \to H^{k+n}(X;\mathbb{Z}/2)$

(b) $Sq^0(x) = x$, $Sq^n(x) = x\smile x$ for all $x \in H^n(X;\mathbb{Z}/2)$

(c) if $x \in H^n(X;\mathbb{Z}/2)$ then $Sq^{n+k}(x) = 0$ for all $k > 0$

(d) (Cartan Formula) $Sq^n(x\smile y) = \sum_{i+j=n} Sq^i(x)\smile Sq^j(y)$.

Recall that the cohomology ring of $\mathbb{R}P^\infty$ is a polynomial ring (actually an algebra) over $\mathbb{Z}/2$ with just one generator $w_1 \in H^1(\mathbb{R}P^\infty;\mathbb{Z}/2)$, the first Stiefel-Whitney class,

$$H^*(\mathbb{R}P^\infty;\mathbb{Z}/2) = \mathbb{Z}/2[w_1, w_1^2, \ldots].$$

The first Steenrod Square is as follows:

- $Sq^1(x) = x^2$
- for $k \geq 1$: $Sq^1(x^{2k}) = Sq^1(x^k)\smile x^k + x^k\smile Sq^1(x^k) = 0$
- for $k > 1$: $Sq^1(x^{2k+1}) = Sq^1(x^{2k})\smile x + x^{2k}\smile Sq^1(x) = x^{2k+2}$.

The second Steenrod operation may be computed using the Cartan Formula for all k (we may not pay attention to signs since computations are in $\mathbb{Z}/2$-coefficients):

$$Sq^2(x^{4k}) = \underbrace{Sq^2(x^{2k})\smile x^{2k} + x^{2k}\smile Sq^2(x^{2k})}_{=0} + \underbrace{Sq^1(x^{2k})}_{=0}\smile\underbrace{Sq^1(x^{2k})}_{=0} = 0,$$

$$Sq^2(x^{4k+1}) = \underbrace{Sq^2(x^{4k})}_{=0}\smile x + x^{4k}\smile\underbrace{Sq^2(x)}_{=0} + \underbrace{Sq^1(x^{4k})}_{=0}\smile Sq^1(x) = 0,$$

$$Sq^2(x^{4k+2}) = \underbrace{Sq^2(x^{4k})}_{=0}\smile x^2 + x^{4k}\smile Sq^2(x^2) + \underbrace{Sq^1(x^{4k})}_{=0}\smile Sq^1(x^2) = x^{4k+4},$$

$$Sq^2(x^{4k+3}) = Sq^2(x^{4k+2})\smile x + x^{4k+2}\smile\underbrace{Sq^2(x)}_{=0} + \underbrace{Sq^1(x^{4k+2})}_{=0}\smile Sq^1(x) = x^{4k+5}.$$

By the properties above we have:

- (for cases $4k$,$4k+1$) Suppose, that $k = 0$. Then

$$Sq^2\colon H^0(\mathbb{R}P^\infty; \mathbb{Z}/2) \to H^2(\mathbb{R}P^\infty; \mathbb{Z}/2),$$

$$Sq^2\colon H^1(\mathbb{R}P^\infty; \mathbb{Z}/2) \to H^3(\mathbb{R}P^\infty; \mathbb{Z}/2),$$

  are zero maps, hence their duals $(Sq^2)^*$ as maps

$$E^2_{2,1} = H_2(\mathbb{R}P^\infty; \mathbb{Z}/2) \to H_0(\mathbb{R}P^\infty; \mathbb{Z}/2) = E^2_{0,2},$$

$$E^2_{3,1} = H_3(\mathbb{R}P^\infty; \mathbb{Z}/2) \to H_1(\mathbb{R}P^\infty; \mathbb{Z}/2) = E^2_{1,2},$$

  has to be zero too. Similarly other zero differentials connecting the first and the second row are marked in red.

  We may now conclude, that

$$E^3_{1,2} = \frac{\ker(d^2\colon E^2_{1,2} \to E^2_{-1,3}) = \mathbb{Z}/2}{\operatorname{im}(d^2\colon E^2_{3,1} \to E^2_{1,2}) = \{1\}} = \mathbb{Z}/2.$$

  Moreover, since all differentials leaving the 2nd row are (obviously) zero their kernel is whole $E^2_{*,2}$ and therefore in cases $* = 4k, 4k+1$ terms on the third page $E^3_{*,2}$ are

$$E^3_{*-2,2} = \frac{\ker\,(d^2\colon E^2_{*-2,2} \to E^2_{*-4,3}) = \mathbb{Z}/2}{\operatorname{im}(d^2\colon E^2_{*,1} \to E^2_{*-2,2}) = \{1\}} = \mathbb{Z}/2$$

  We may also observe that

$$E^3_{2,1} = \frac{\ker(d^2\colon E^2_{2,1} \to E^2_{0,2}) = \mathbb{Z}/2}{\operatorname{im}(d^2\colon E^2_{3,0} \to E^2_{2,1}) = \{1\}} = \mathbb{Z}/2.$$

- In cases $4k+2, 4k+3$ second Steenrod squares $H^{4k+2}(\mathbb{R}P^\infty; \mathbb{Z}/2) \to H^{4k+4}(\mathbb{R}P^\infty; \mathbb{Z}/2)$ are isomorphisms, hence theirs duals

$$(Sq^2)^*\colon E^2_{4k+4,1} = H_{4k+4}(\mathbb{R}P^\infty; \mathbb{Z}/2) \to H_{4k+2}(\mathbb{R}P^\infty; \mathbb{Z}/2) = E^2_{4k+2,2},$$

$$(Sq^2)^*\colon E^2_{4k+5,1} = H_{4k+5}(\mathbb{R}P^\infty; \mathbb{Z}/2) \to H_{4k+3}(\mathbb{R}P^\infty; \mathbb{Z}/2) = E^2_{4k+3,2},$$

  also are isomorphisms.

  Finally it follows that $E^3_{*,2} = \bullet$ for $* = 4k+2, 4k+3$.

- To finish the calculation we need to know the last unknown group on the 4-th diagonal on E^3, $E^3_{3,1}$. Hence we need to compute the image of d^2 differential

$$d^2\colon E^2_{5,0} \to E^2_{3,1}.$$

  The mod 2 coefficient reduction homomorphism (call it ϱ_2) in case of $\mathbb{R}P^\infty$ yields an isomorphism in odd dimensions (tip: observe, that in the cellular complex computing cohomology of $\mathbb{R}P^\infty$ with $\mathbb{Z}/2$ coefficients all differentials become 0), hence the composition $Sq^2 \circ \mathbb{Z}/2$ is an isomorphism if and only if Sq^2 is. Thus the differential

$$(Sq^2)^* \circ \varrho_2 \colon H_*(\mathbb{R}P^\infty;\mathbb{Z}) = E^2_{*,0} \to E^2_{*-2,1} = H_{*-2}(\mathbb{R}P^\infty;\mathbb{Z}/2)$$

is an isomorphism if $* = 4k+1$ and 0 in other cases (recall that all even entries in 0th row are 0). In particular the yellow differential is an isomorphism and

$$E^3_{3,1} = \frac{\ker(d^2 \colon E^2_{3,1} \to E^2_{1,2}) = \mathbb{Z}/2}{\operatorname{im}(d^2 \colon E^2_{5,0} \to E^2_{3,1}) = \mathbb{Z}/2} = \bullet.$$

Noting that the 4-th diagonal stabilizes on the E^3 page we can finally read the answer aloud

$$\Omega_4^{Spin}(K(\mathbb{Z}/2,1)) = \mathbb{Z}$$

(coming from the *Spin* bordism of a point).

(e) and (f) case $r=1$: we compute $\Omega_6^{Spin}(K(\mathbb{Z},2))$ up to extension and we show that $\Omega_7^{Spin}(K(\mathbb{Z},2)) = 0$.

Now we use the same tricks applied to the same spectral sequence. The only difference is that

$$K(\mathbb{Z},2) \cong \mathbb{C}P^\infty,$$

hence we have (co)homology only in even dimensions.

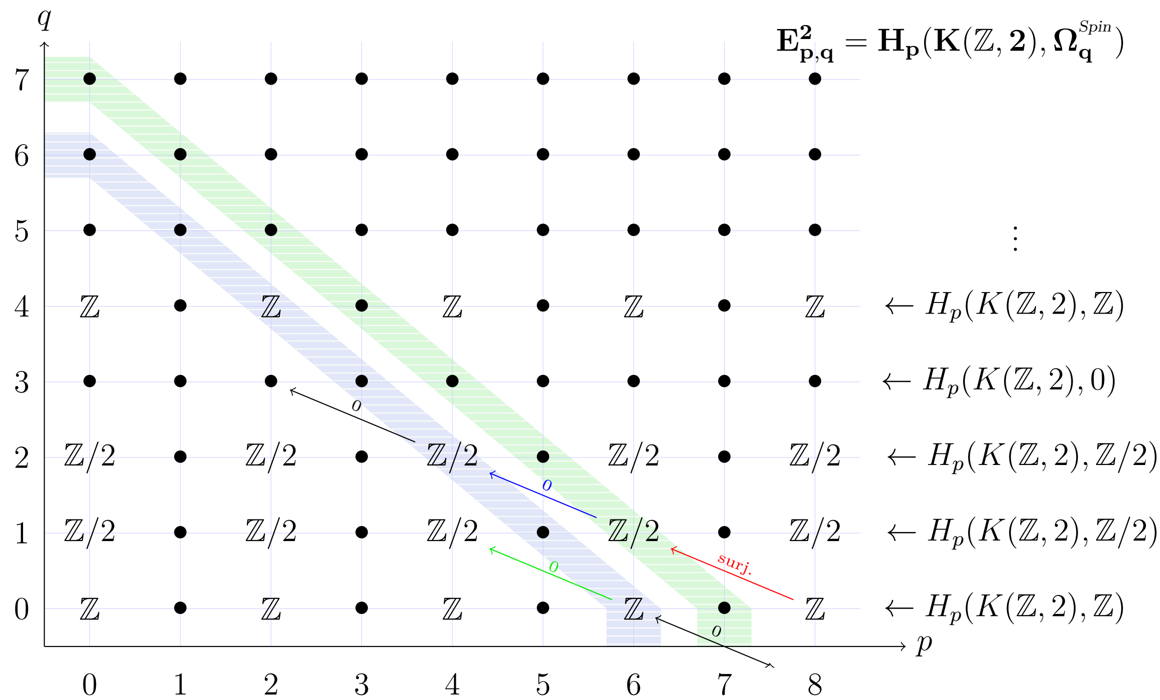
Let us begin with short revision of homology and cohomology of $K(\mathbb{Z},2)$:

$$H_p(K(\mathbb{Z},2);\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } p \text{ even,} \\ 0 & \text{for } p \text{ odd,} \end{cases}$$

$$H_p(K(\mathbb{Z},2);\mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & \text{for } p \text{ even,} \\ 0 & \text{for } p \text{ odd,} \end{cases}$$

$$H^*(K(\mathbb{Z},2);\mathbb{Z}/2) = \mathbb{Z}/2[x, x^2, \ldots], \text{ where } x \in H^2(K(\mathbb{Z},2);\mathbb{Z}/2)$$

After plugging in the above results our $E^2_{p,q} = H_p(K(\mathbb{Z},2);\Omega_q^{Spin}(\mathrm{pt}))$ page of the spectral sequence looks as follows.



We are interested only in computing $\Omega^{Spin}_6(X)$ and $\Omega^{Spin}_7(X)$ for $X = K(\mathbb{Z}, 2)$, so we will be looking at the 6th and the 7th diagonal. Hence it suffices to (and we will) compute just the three coloured differentials, since all higher differentials entering 6th or 7th diagonal will be trivial. We we will also give some more general statements allowing to compute all differentials connecting 0th, 1st and 2nd row.

- The $d^2\colon E^2_{6,1} \to E^2_{4,2}$ differential is the dual of

  $$Sq^2\colon H^4(K(\mathbb{Z}, 2); \mathbb{Z}/2) \to H^6(K(\mathbb{Z}, 2); \mathbb{Z}/2).$$

  Since $H^4(K(\mathbb{Z}, 2); \mathbb{Z}/2)$ is generated by x^2 we need to compute $Sq^2(x^2)$. Observe that Sq^1 (or more generally: Sq^{odd}) vanish in the cohomology ring of $K(\mathbb{Z}, 2)$ because the 1st gradation (or any odd gradation) group vanish. Then the evaluation of Steenrod square on the generator yields

  $$Sq^2(x^2) = 2(Sq^2(x)\smile x) + Sq^1(x)\smile Sq^1(x) = 0$$

  and therefore $(Sq^2(x^3))^* = (Sq^2)^*(a) = d^2(a) = 0$ where a generates $H^3(K(\mathbb{Z}, 2); \mathbb{Z}/2)$. This gives us

  $$E^3_{4,2} = \mathbb{Z}/2$$

  since the differential leaving $E^2_{4,2}$ is obviously 0.

- Since the differential entering $E^2_{6,0}$ is obviously 0, hence $E^3_{6,0}$ is the kernel of the differential

  $$d^2\colon E^2_{6,0} \to E^2_{4,1}.$$

The differential is given as the dual of

$$Sq^2\colon H^4(K(\mathbb{Z},2);\mathbb{Z}/2) \to H^6(K(\mathbb{Z},2);\mathbb{Z}/2)$$

composed with the $\mathbb{Z}/2$-coefficient reduction homomorphism ϱ_2. However, regardless what ϱ_2 is, $Sq^2(x^2) = 0$ hence $(Sq^2)^*(x^3) = 0$ and $d^2 = 0 \circ \varrho_2 = 0$. Thus $E^3_{6,0} = \mathbb{Z}$.

- The $d^2\colon E^2_{8,0} \to E^2_{6,1}$ differential is the dual of

$$Sq^2\colon H^6(K(\mathbb{Z},2);\mathbb{Z}/2) \to H^8(K(\mathbb{Z},2);\mathbb{Z}/2)$$

composed with the $\mathbb{Z}/2$-coefficient reduction homomorphism ϱ_2.
The Steenrod square evaluated on x^3 (the generator of $H^6(K(\mathbb{Z},2);\mathbb{Z}/2)$) is

$$Sq^2(x^3) = Sq^2(x^2)\smile x + x^2\smile Sq^2(x) = x^2\smile x^2 = x^4.$$

Since x^4 generates $H^4(K(\mathbb{Z},2);\mathbb{Z}/2)$ hence Sq^2 is an isomorphism. More generally $Sq^2(x^{2k+1}) = Sq^2(x^{2k})\smile x + x^{2k}\smile Sq^2(x) = x^{2k+2}$.
Observe that the $\mathbb{Z}/2$-coefficient reduction homomorphism

$$\varrho_2\colon H^k(K(\mathbb{Z},2);\mathbb{Z}) \to H^k(K(\mathbb{Z},2);\mathbb{Z}/2)$$

is in our case always surjective, hence

$$d^2 = (Sq^2)^* \circ \varrho_2\colon E^2_{8,0} \to E^2_{6,1}$$

is surjective and has kernel $2\cdot\mathbb{Z}$.
This allows us to compute

$$E^3_{6,1} = \frac{\ker(d^2\colon E^2_{6,1} \to E^2_{4,2}) = \mathbb{Z}/2}{\mathrm{im}(d^2\colon E^2_{8,0} \to E^2_{6,1}) = \mathbb{Z}/2} = \bullet.$$

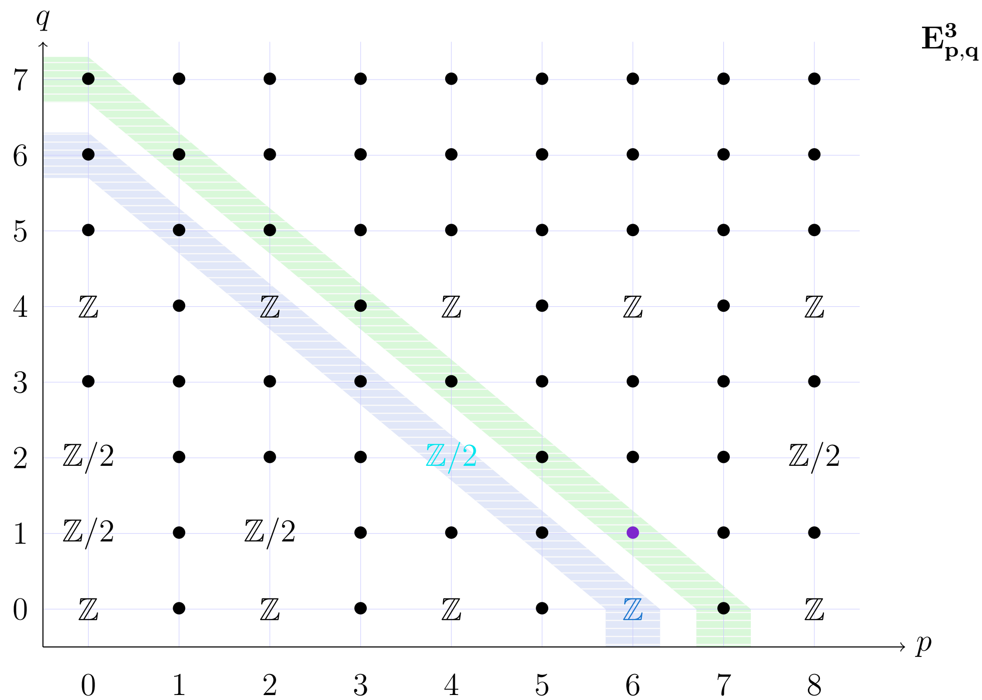
(Of course we didn't need to compute the kernel above, information on surjectivity of the differential in the numerator would suffice.)

More general, if $Sq^2\colon H^{*-2}(K(\mathbb{Z},2);\mathbb{Z}/2) \to H^*(K(\mathbb{Z},2);\mathbb{Z}/2)$ is an isomorphism (what happens for $* = 4p$) then

$$d^2 = (Sq^2)^* \circ \varrho_2\colon E^2_{*,0} \to E^2_{*-2,1}$$

is surjective and (if non-trivial) has the same kernel as ϱ_2 (which is $2\cdot\mathbb{Z}$).

Although it is not needed to complete our task checking that the E^3-page looks as follows is a relaxing exercise.



Although we can clearly read the answer that all spin 7-manifolds are $K(\mathbb{Z},2)$-bordant to a point: $\Omega_7^{Spin}(K(\mathbb{Z},2)) = 0$). For $\Omega_6^{Spin}(K(\mathbb{Z},2))$ we have an extension problem to solve. In fact the extension is non-trivial. For a proof see [F, Proposition 1.1].

8. No solution.

9. No solution.

2 Tuesday

1. (Inanc Baykur)

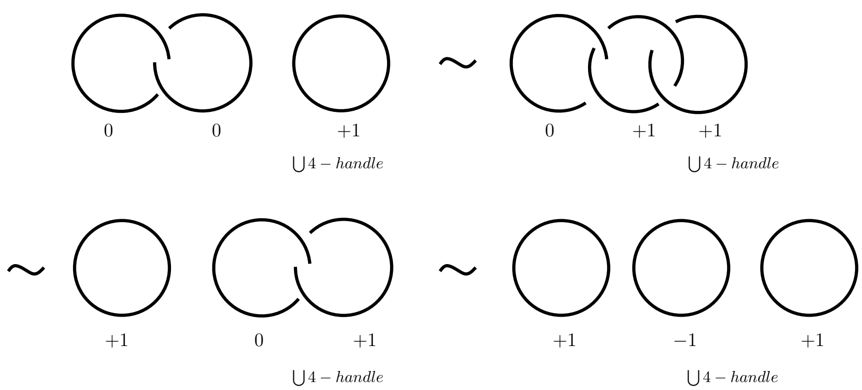
   Here, I would like to demonstrate how various techniques, which were briefly mentioned during the summer school, can be employed to get sharper (and sharper) results. Below, we will not only answer the original question asking which one of the given pairs of 4-manifolds are stably or strictly stably diffeomorphic, but will also determine the minimum number of stabilizations required for each pair, including the case when the two manifolds are diffeomorphic in the first place. Recall that Diarmuid defined "strict" stable diffeomorphism as the one where both manifolds are stabilized the same number of times.

   First of all, $S^2 \times S^2$, $\mathbb{CP}^2$, $\overline{\mathbb{CP}}^2$, and $\overline{\mathrm{K3}}$ are all simply-connected closed oriented 4-manifolds, so are their connected sums —where the bar indicates the orientation reversal. Moreover, each one of the manifolds given in the problem contains at least one $\mathbb{CP}^2$ or $\overline{\mathbb{CP}}^2$ summand, so there is a second homology class with odd self-intersection (i.e. ± 1), implying that all these manifolds are non-spin. Therefore, by the theorem Matthias discussed in his lecture, each pair of 4-manifolds we are given would be strictly stably diffeomorphic if and only if their euler characteristics and signatures agreed. Simple calculation shows that signatures do agree, which are 2, 17, and 15 for the pairs in (a), (b), and (c), respectively, and the euler characteristics of the pairs in (a) and (c) already agree, whereas one needs to stabilize the $17\mathbb{CP}^2$ in part (b) three times to match the euler characteristics of it to that of $\overline{\mathrm{K3}}\#\mathbb{CP}^2$.

We conclude that the manifolds in (a) and (c) are strictly stably diffeomorphic, and the ones in (b) are stably diffeomorphic, answering the original question.

So far all we know is that after some number of stabilizations each pair of manifolds in (a)-(c) will become diffeomorphic. Now we can try to determine the minimum number of stabilizations needed for each pair. In (b), we clearly need to at least stabilize the second manifold three times, so let us instead answer this question for (b') $\overline{\mathrm{K3}}\#\mathbb{CP}^2$ and $17\mathbb{CP}^2\#3(S^2 \times S^2)$.

We will use the following lemma repeatedly: $S^2 \times S^2\#\mathbb{CP}^2 = 2\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$, and $S^2 \times S^2\#\overline{\mathbb{CP}}^2 = \mathbb{CP}^2\#2\overline{\mathbb{CP}}^2$.



A proof of this lemma can be given by simple handle calculus. Consider the handlebody on $S^2 \times S^2$ given by only two 2-handles (so without any 1- or 3-handles), which is given by a Hopf link whose both components have framing zero. Connect summing with $\mathbb{CP}^2$ amounts to adding a (+1)-framed unknot to this diagram, over which one can then slide one of the two 0-framed 2-handles, getting three unknots with framings 0, +1, and +1, linked with each other as shown in the figure. One can then unlink one of the (+1)-framed handles and isotope it away after sliding it over the 0-framed 2-handle. The remaining Hopf link with framings 0 and +1 is then equivalent to two unknots with framings −1 and +1, which can be easily seen for example by considering the handle slide in the opposite direction (sliding a (−1)-framed unknot over a (+1)-framed unknot. (Observe that this shows that the twisted S^2-bundle over S^2 has the total space diffeomorphic to $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$.) It is easy to check that the same proof works verbatim when we take connect sums with $\overline{\mathbb{CP}}^2$ instead of $\mathbb{CP}^2$.

The lemma implies that the 4-manifolds in (a) are already diffeomorphic.

For parts (b') and (c), let me first show that one stabilization is enough for both. A nice way to see this would follow from a 5-dimensional cobordism argument (after

Mandelbaum) for expressing K3 as the fiber sum of two copies of $E(1) = \mathbb{CP}^2\#9\overline{\mathbb{CP}}^2$ along regular torus fibers of the elliptic fibrations on each, as mentioned in Zoltan's lecture. The cobordism W between two disjoint copies of $E(1)$ and K3 prescribed by the fiber sum operation is given by attaching $I \times D^2 \times T^2$ along the tubular neighborhoods of the tori in each copy of $E(1)$. The standard handle decomposition of T^2 as one 0-handle $h^{(0)}$, two 1-handles $h_1^{(1)}$ and $h_2^{(1)}$, and one 2-handle $h^{(2)}$, gives rise to expressing this 5-dimensional cobordism as

$$W = I \times (E(1) \sqcup E(1)) \cup I \times h^{(0)} \times D^2 \cup I \times h_1^{(1)} \times D^2 \cup I \times h_2^{(1)} \times D^2 \cup I \times h^{(2)} \times D^2.$$

Taking the product of the handles of T^2 with the interval I and then thickening by D^2, we get 5-dimensional handles of one higher index. After attaching the 1-handle to the disjoint union $E(1) \sqcup E(1)$, what we get is nothing but $E(1)\#E(1) = 2\mathbb{CP}^2\#18\overline{\mathbb{CP}}^2$. Adding the two 2-handles has the same effect as connect summing with two copies of $S^2 \times S^2$, since the manifold we attach these handles to is simply-connected (so the attaching circle of the 2-handle can be isotoped to a circle in a standard 4-ball, and therefore the handle attachment yields to connect summing with $S^2 \times S^2$ or $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$) and non-spin (so we can modify the framing of the attached 2-handle to get either $S^2 \times S^2$ or $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$ as we like). What is left is the 3-handle, which upside down is a 2-handle attached to the simply-connected 4-manifold K3. So the level of the cobordism W before the attachment of the 3-handle is equal to $2\mathbb{CP}^2\#18\overline{\mathbb{CP}}^2\#2(S^2 \times S^2) = 4\mathbb{CP}^2\#20\overline{\mathbb{CP}}^2$ from one end and to K3 connect summed with $S^2 \times S^2$ or $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$ from the other end. As remarked above, the ambiguity with the framing disappears when we introduce one more $\mathbb{CP}^2$ or $\overline{\mathbb{CP}}^2$ summand since we then get a simply-connected non-spin 4-manifold.

We conclude that both K3$\#\mathbb{CP}^2$ and K3$\#\overline{\mathbb{CP}}^2$, after one stabilization with $S^2 \times S^2$, become diffeomorphic to a connect sum of $\mathbb{CP}^2$s and $\overline{\mathbb{CP}}^2$s, which of course holds true when we replace K3 by $\overline{\text{K3}}$ as well. Hence, understanding the handle structure of a certain cobordism as above and some baby handle calculus give us a real sharp answer: We now know that the minimum number of stabilizations needed in (a) is zero, and it is at most one in (b') and in (c).

We can in fact tell whether the minimum number of stabilizations needed in (b') and (c) are 0 or 1, but to do so, we will need to switch gears, and employ Gauge theory and advanced Kirby calculus.

For part (b'), we claim that the manifolds $\overline{\text{K3}}\#\mathbb{CP}^2$ and $17\mathbb{CP}^2\#3(S^2 \times S^2) = 20\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2$ in (b') are not diffeomorphic. Let me use Seiberg-Witten invariants (SW in short) to argue this, though any one of the celebrated smooth invariants (such as Donaldson or Ozváth-Szabó invariants) can be used to run the same arguments. The SW invariant of the K3 surface it nontrivial, and after blow-ups one still gets nontrivial SW invariants. However, it is known that if $X = X_1\#X_2$ with $b^+(X_i) > 0$ for each $i = 1, 2$, then SW invariant of X is trivial. In particular SW invariant of $3\mathbb{CP}^2\#20\overline{\mathbb{CP}}^2$ is trivial. So K3$\#\overline{\mathbb{CP}}^2 \neq 3\mathbb{CP}^2\#20\overline{\mathbb{CP}}^2$, and thus, the pair of 4-manifolds we have got in (b') which we get after orientation reversal, cannot be diffeomorphic. Going back to the original problem; the minimum number of stabilizations needed for the 4-manifolds in (b) to become diffeomorphic are 1 and 4, respectively.

For part (c) on the other hand, we claim that the minimum number is zero. This follows from the fact that K3 is almost completely decomposable, that is, after blowing-up once, it becomes diffeomorphic to a connect sum of $\mathbb{CP}^2$ and $\overline{\mathbb{CP}}^2$s by a theorem of Moishezon. This can be verified by a more involved handle calculus, too, which can be found in the book of Gompf and Stipsicz —I will not include a proof here so as to avoid taking up much more space. Once again, reversing the orientations, we see that the pair of 4-manifolds we are given are diffeomorphic in the first place.

Hopefully this step-by-step advancing solution promotes faith in the fellowship of surgery theory, handle calculus, and Gauge theory in the study of smooth 4-manifolds.

2. (Ahmet Beyaz)

(2a) For $j = 0$ and $j = 1$, $\pi_1(M_j)$ and $\pi_2(M_j)$ are trivial. For the stably diffeomorphism result, we need normal type for M_j. Consider the Gauss map of an embedding of M_j, $\nu_{M_j} : M_j \to BSO$. BSO has a 2-fold cover BSpin and ν_{M_j} has a lift to this cover $\overline{\nu_{M_j}}$. $\pi_1(BSO)$ is trivial, but $\pi_2(BSO)$ is not. But we take B as BSpin whose second homotopy group is trivial. And whatever the fiber of the fibration $B \to BSO$, its higher degree homotopy is trivial. So $\overline{\nu_{M_j}}$ is a 3-equivalence.

The bordism group of spin 6-manifolds, Ω_6^{Spin} is trivial, therefore the manifolds M_0 and M_1 are spin bordant, hence bordant over their normal 2-type. Then by a theorem from Tuesday's lectures, [K, Corollary 3], M_0 and M_1 are stably equivalent. i.e. there is r and s such that

$$M_0\#_r(S^3 \times S^3) \text{ is diffeomorphic to } M_1\#_s(S^3 \times S^3).$$

Now if they are strictly stably diffeomorphic then $r = s$;
$\Rightarrow M_0\#_r(S^3 \times S^3)$ is diffeomorphic to $M_1\#_r(S^3 \times S^3)$
$\Rightarrow e(M_0\#_r(S^3 \times S^3)) = e(M_1\#_r(S^3 \times S^3))$
$\Rightarrow e(M_0) = e(M_1)$.
If $e(M_0) = e(M_1)$ then $e(M_0\#_r(S^3 \times S^3)) = e(M_1\#_s(S^3 \times S^3))$ which means that $r = s$.

(2b) If $r = 0$ then take S^6 as M. If $r > 0$ take $\#_r(S^3 \times S^3)$ as M.

(2c) If $M_0\#_r(S^3 \times S^3)$is diffeomorphic to$M_1\#_r(S^3 \times S^3)$ and If W is an h-cobordism then $\#_r(S^3 \times S^3)$ can be cancelled using the fact that $l_7(e) \cong \mathcal{E}l_7(e)$, [C-S, Proposition 6.20], and by the h-cobordism theorem M_0 and M_1 are diffeomorphic; equivalently one can use [K, Proposition 7.4]. The other direction is trivial.

3. No solution.

4. (Raphael Zentner and DC)

(a) We first show that that $H^3(B\mathrm{Spin}^{\mathbb{C}};\mathbb{Z}) = 0$. Observe that $\mathrm{Spin}(n)$ is a subgroup of $\mathrm{Spin}^{\mathbb{C}}(n)$ and that there is an induced fibration sequence

$$\mathrm{Spin} \to \mathrm{Spin}^{\mathbb{C}} \to S^1.$$

Now for any topological group G with subgroup $H \subset G$ there is a fibration sequence $G/H \to BH \to BG$ and so we have a fibration sequence

$$S^1 \to B\mathrm{Spin} \to B\mathrm{Spin}^{\mathbb{C}}.$$

From the fact that $B\mathrm{Spin}$ is 3-connected we deduce from the integral cohomology Leray-Serre spectral sequence (see [W, Ch. XIII Theorem 4.9*]) of the above fibration that $H^3(B\mathrm{Spin}^{\mathbb{C}};\mathbb{Z}) = 0$.

There are models for the classifying space functor which take short exact sequences of topological groups to fibration sequences of spaces: this follow easily from [S, Theorem 7.6]. Hence, as stated in the problem, we have a fibration sequence

$$BS^1 \to B\mathrm{Spin}^{\mathbb{C}} \to BSO. \tag{1}$$

Since $BS^1 \simeq K(\mathbb{Z}, 2)$ is an Eilenberg MacLane space, the sole obstruction to lifting the identity map $\mathrm{Id}\colon BSO \to BSO$ to a section of the fibration $B\mathrm{Spin}^{\mathbb{C}} \to BSO$ is an obstruction $\alpha \in H^3(BSO;\mathbb{Z})$: see [W, Ch. VI Corollary 6.14]. From the fact above that $H^3(B\mathrm{Spin}^{\mathbb{C}};\mathbb{Z}) = 0$ and from the integral cohomology Leray-Serre spectral sequence for the fibration (1) we deduce that α the transgression homomorphism $t\colon H^2(BS^1;\mathbb{Z}) \to H^3(BSO;\mathbb{Z})$ must be non-zero and hence α must be non-zero. On the other hand by [M-S, Theorem 7.1 & Theorem 15.9] we have that

$$H^*(BSO;\mathbb{Z}/2) \cong \mathbb{Z}/2[w_2, w_3, \dots] \quad \text{and} \quad H^*(BSO;\mathbb{Q}) \cong \mathbb{Q}[p_1, p_2, \dots].$$

It follows that $H^3(BSO;\mathbb{Z}) \cong \mathbb{Z}/2(\alpha)$ and that we may identify α with βw_2 where $\beta\colon H^2(BSO;\mathbb{Z}/2) \to H^3(BSO;\mathbb{Z})$ is the Bockstein homomorphism associated to the short exact coefficient sequence $\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2$.

Now suppose that $\tau\colon M \to BSO$ classifies the stable tangent bundle of the oriented manifold M. From the discussion above we know that M admits a $\mathrm{Spin}^{\mathbb{C}}$ structure if and only if $0 = \tau^*(\beta w_2) = \beta w_2(M) \in H^3(M;\mathbb{Z}/2)$. But by definition $\beta w_2(M) = 0$ if and only if $w_2(M)$ has an integral lift in $H^2(M;\mathbb{Z})$.

Finally, for any manifold X, the total Stiefel-Whitney classes of the stable tangent bundle of X and the stable normal bundle of X are related by the formula

$$w(X) \cdot w(\nu_X) = 1 \in H^*(X;\mathbb{Z}/2).$$

Since M is orientable, $w_1(M) = 0 = w_1(\nu_M)$ and so $w_2(M) = w_2(\nu_M)$. Thus M admits a normal $\mathrm{Spin}^{\mathbb{C}}$ structure if and only if M admits a stable tangential $\mathrm{Spin}^{\mathbb{C}}$ structure if and only if $w_2(M)$ admits an integral lift.

(b) We show that every closed oriented 4-manifold M admits a normal Spin$^{\mathbb{C}}$ structure. By part (a) this is true if and only if $w_2(M)$ admits an integral lift. First we do this under the assumption that $\pi_1(M)$ is a finite group of odd order. Associated to the short exact sequence of groups

$$0 \to \mathbb{Z} \stackrel{\times 2}{\to} \mathbb{Z} \to \mathbb{Z}/2 \to 0$$

is the following part of the long exact sequence in cohomology:

$$\cdots \to H^2(M;\mathbb{Z}) \stackrel{\times 2}{\to} H^2(M;\mathbb{Z}) \stackrel{r}{\to} H^2(M;\mathbb{Z}/2) \stackrel{\beta}{\to} H^3(M;\mathbb{Z}) \stackrel{\times 2}{\to} H^3(M;\mathbb{Z}) \to \dots . \tag{2}$$

Here r is the reduction morphism and β is the Bockstein operator. It is enough to show that the map $\times 2$ following β in this sequence is injective. Because Poincaré duality is natural this map is injective if and only if the map

$$H_1(M;\mathbb{Z}) \stackrel{\times 2}{\to} H_1(M;\mathbb{Z})$$

is injective. The result follows if $H_1(M;\mathbb{Z})$, which is the abelianisation of $\pi_1(M)$, has no 2-torsion.

To show this, assume there is an element $\bar{x} \in H_1(M;\mathbb{Z})$ such that $2\bar{x} = 0$. Let $x \in \pi_1(M)$ be a lift of $\bar{x}$. The assumption implies that x^2 lies in the commutator subgroup $[\pi_1(M), \pi_1(M)]$ of $\pi_1(M)$. If the groups $\langle x \rangle$ and $\langle x^2 \rangle$ generated by x and x^2 were distinct, it would follow from Lagrange's theorem, applied to the inclusion of subgroups

$$\langle x^2 \rangle \subseteq \langle x \rangle \subseteq \pi_1(M) ,$$

that the order of $\pi_1(M)$ would be divisible by 2. As this is not the case we conclude $\langle x^2 \rangle = \langle x \rangle$, but therefore x also lies in the commutator subgroup, and so $\bar{x} = 0$.

We sketch the general argument, following the original source [H-H]. Observe that there are two submodules in $H^2(M;\mathbb{Z}/2)$, namely the images $r(T)$ and $r(H^2(M;\mathbb{Z}))$ of the torsion subgroup $T \subseteq H^2(M;\mathbb{Z})$ and the full cohomology group $H^2(M;\mathbb{Z})$, under the restriction map r from the sequence above. We have the inclusions

$$r(T) \subseteq r(H^2(M;\mathbb{Z})) \subseteq H^2(M;\mathbb{Z}/2) .$$

One considers the cup product pairing followed by evaluation on the fundamental cycle

$$\begin{aligned} q :& H^2(M;\mathbb{Z}/2) \times H^2(M;\mathbb{Z}/2) \to \mathbb{Z}/2 \quad , \\ & (x, y) \mapsto \langle x \smile y, [M] \rangle \ . \end{aligned}$$

By Poincaré duality with $\mathbb{Z}/2$ coefficients this pairing is a non-degenerate bilinear form over the vector space $H^2(M;\mathbb{Z}/2)$. Since M is Spin$^{\mathbb{C}}$ and so oriented, the reduction modulo 2 of the integer pairing $\langle x \smile y, [M] \rangle$ is equal to the above pairing

$q(r(x), r(y))$. Thus q restricted to $r(T)$ is zero and more generally $q(r(x), r(y)) = 0$ for all elements $x \in T$ and $y \in H^2(M;\mathbb{Z})$. Now Hirzebruch and Hopf claim that more is true: $r(T)$ and $r(H^2(M;\mathbb{Z}))$ are mutual annihilators of q, meaning that, if

$$q(x, r(t)) = 0$$

for all $t \in T$, then x must belong to $r(H^2(M;\mathbb{Z}))$, and analogously for the converse statement. We prove this now: write $H^2(M;\mathbb{Z}) = T \oplus F$ where F is a finitely generated free abelian group where $r(F) = F/2F$ is a summand of $\mathbb{Z}/2$ which on which q is non-singular. It follows that there is an orthongal decomposition

$$(H^2(M;\mathbb{Z}/2), q) \cong (F/2F, q|_{F/2F}) \oplus (V, q_V).$$

We see that $r(T) \subset V$ and inspecting the Bockstein sequence (2) above we see that $\mathrm{rank}_{\mathbb{Z}/2}(V) = 2\mathrm{rank}_{\mathbb{Z}/2}(r(T))$. Since $r(T)$ is annihilated by q we see that (V, q) must be isomorphic to the hyperbolic form on $r(T)$. In particular, the annilhiator of $r(T)$ in $(H^2(M;\mathbb{Z}/2), q)$ is precisely $r(T) \oplus F/2F$ and this is precisely $r(H^2(M;\mathbb{Z}))$.

To prove that $w_2(M)$ admits an integral lift one needs one further piece of knowledge: It turns out that $w_2(M)$ is dual to the Steenrod square $Sq^2 : H^2(M;\mathbb{Z}/2) \to H^4(M;\mathbb{Z}/2)$ (which is just the usual cup product squaring in this situation), which was proved by Wu. To be more precise, it follows from the non-degeneracy of the form q that there is an element $v_2 \in H^2(M;\mathbb{Z}/2)$ such that

$$Sq^2(x) = x \smile x = x \smile v_2(M) \tag{3}$$

for all $x \in H^2(M;\mathbb{Z}/2)$. Since M is orientable $v_1(M) = w_1(M) = 0$ and so $v_2(M) = w_2(M)$ by the Wu-formula: See [M-S, Theorem 11.14].

Now the proof follows easily. From equation (3) it follows that

$$q(w_2(M), x) = \langle x \smile x, [M]\rangle = 0$$

for all elements $x \in r(T)$. By the statement of the mutual annihilators one therefore concludes that $w_2(M) \in r(H^2(M;\mathbb{Z}))$.

5. (Khaled Qazaqzeh) The classification of such bilinear forms is a theorem due to Serre:

   Theorem 2.1. *Two indefinite unimodular symmetric bilinear forms over $\mathbb{Z}$ are isometric if and only if they have the same rank, type and signature.*

6. No solution.

7. No solution.

8. (a) In computing the first homology of the surgered 3-manifold, we appeal to the Mayer-Vietoris sequence. The role of one subspace is played by (an open neighbourhood of) the complement of the knot, while the other one is by the solid torus we will glue. The

homology of the intersection (being an open tubular neighbourhood of a 2-dimensional torus) is $\mathbb{Z}\oplus\mathbb{Z}$, and one generator can be chosen to be the longitude λ of the knot (which is homologically trivial in the knot complement) and the other one by the meridian μ of the knot. Since the complement of the knot has first homology generated by the meridian of the knot, and for the surgery coefficients ± 1 we attach the solid torus back in such a way that $\lambda \pm \mu$ will bound a disk, it follows that the surgered manifold has vanishing first homology (with integral coefficients). This means that the 3-manifold is an integral homology sphere.

(b) Regarding D^4 as a 0-cell and the 2-handle as a thickened 2-cell, we see that the 4-manifold is homotopy equvalent to a CW-complex with a single 0- and a single 2-cell, which is exactly S^2. In order to compute the intersection form, fix a Seifert surface Σ of the knot K. Consider a push-off K^+ of K along the given framing and notice that K^+ intersects Σ in k points (with equal sign, dictated by the sign of the framing). Now take a Seifert surface Σ^+ of K^+ and push its interior into D^4. When we attach the 2-handle, we can choose the attaching map in such a way that the core disk glues to Σ (providing a closed surface in $W(K, k)$. The chosen framing implies that a parallel copy of the core disk can be arranged to glue to Σ^+. This procedure then provides two surfaces, both providing a fixed generator of the second homology of $W(K, k)$, intersecting each other transversally in k points with the sign dictated by the framing of the surgery. This construction then verifies the statement.

9. (Andrew Donald)

Let $L = K_1 \cup K_2 \cup \ldots \cup K_r$ be a framed link in S^3 with framing $k = (k_1, \ldots, k_r)$. Denote the 3-manifold produced by surgery on this link by Y and, as in the previous question, let $W = W(L, k)$ be 4-manifold with boundary Y obtained by gluing D^4 to the trace of the surgery.

To generalise the previous exercise there are three points which need to be shown – that the homology of Y is determined by L; that the intersection form of W is also determined by L and that W has the homotopy type of a wedge of r copies of S^2.

It is convenient to consider these in the reverse order.

The 4-manifold W is formed by attaching a 2-handle to D^4 along each component of L. Up to homotopy, an n-handle is the same as an n-cell so W has the same homotopy type as a wedge of r 2-spheres – the result of attaching r 2-cells to a 0-cell.

Next, consider the intersection form of W. It is apparent that $H^2(W) \cong \mathbb{Z}^r$ with a generator x_i corresponding to the component K_i of L. It is easy to find a surface representative for these generators by taking a spanning surface for each component in S^3, pushed into D^4, along with the core disc of the 2-handle attached along it. Denote these surfaces by F_i. The intersection form of W with respect to this basis is given by the matrix of linking numbers of the components K_i and the framings. To do this, we should choose an orientation for the link and the induced one for the surfaces.

The same argument as applied in the previous shows that this gives the correct intersection number for a generator with itself is given by the framing. The value of

$Q_W(x_i, x_j)$ for $i \neq j$ is the intersection number of F_i and F_j. View D^4 as the ball of radius 1. Choosing some $0 < t < 1$, it can be arranged that the surface F_i is embedded in such a way as to sit outside the ball of radius t. A surface F'_j can be obtained by projecting the knot K_j radially inwards onto the boundary of this ball of radius t and then pushing a spanning surface inside. The intersections of F_i and F'_j then correspond to intersections of K_j and a Seifert surface for K_i. This is the linking number.

All that now remains is to determine the homology of Y. Due to Poincaré duality it will be sufficient to compute just $H_1(Y) \cong H^2(Y)$.

Given the canonical basis of $H_2(W)$ generated by the 2-handles, there are canonical bases of $H^2(W)$ and $H^2(W, Y)$ given by taking the duals and Poincaré duals respectively.

It is a useful fact that, with respect to these bases, the natural map $j; H^2(W, Y) \to H^2(W)$ is given by Q_W. This is because Q_W represents the cup product pairing on $H^2(W, Y)$. It is evaluated by taking a pair (a, b) and evaluating $a \cup b$ on the fundamental class but this is the same as taking the Poincaré dual of a and evaluating $j(b)$ on this cycle. Applying this to the basis elements verifies that j is represented by Q_W.

Since W is simply connected, the exact sequence for cohomology is

$$\to H^2(W, Y) \to H^2(W) \to H^2(Y) \to 0.$$

This identifies $H^2(Y)$ with $\operatorname{coker} Q_W$. In particular it has infinite order only when Q_W has determinant zero and has order $|\det Q_W|$ otherwise.

10. (Ju A Lee)

Since connecting sum with $S^2 \times S^2$ doesn't change the boundary, it's enough to find any pair K_0, K_1 producing nondiffeomorphic boundaries, $\chi(K_0, 1)$ and $\chi(K_1, 1)$. If we take $K_0 =$ unknot and $K_1 =$ trefoil, then $\chi(K_0, 1) = S^3$ is nonhomeomorphic to the poincare homology sphere $\chi(K_1, 1)$.

3 Thursday

Let D be a link in S^3 equipped with the following extra structure: there is a sublink $D_1 \subset D$ which is the unlink, and each component of D_1 is decorated by a dot, while components of $D - D_1$ are decorated by integers. Such a decorated diagram D is called a *Kirby diagram*. It determines a 4-manifold $X(D)$ with boundary $Y(D) = \partial X(D)$ as follows.

Consider the spanning disks of the dotted circles in D_1, push them into D^4 and delete an open tubular neighbourhood of each disk in D^4. The result will be diffeomorphic to the boundary connected sum of $m = |D_1|$ copies of $S^1 \times D^3$ (the 4-manifold we get by attaching m 1-handles to D^4). To complete the construction of $X(D)$, attach 4-dimensional 2-handles along the knots in $D - D_1$ with the attached integer as framing.

If $Y(D)$ is diffeomorphic to the connected sum of n copies of $S^1 \times S^2$, then indeed $X(D)$ can be completed to a closed 4-manifold $Z(D)$ by attaching n 3-handles and a 4-handle to $X(D)$. It follows from a result of Laudenbach and Poénaru that the diffeomorphism type of $Z(D)$ is independent of how these 3- and 4-handles are attached, and hence is determined by $X(D)$ and ultimately by D. It follows from simple Morse theory that any closed 4-manifold W admits a presentation as $Z(D)$ for some Kirby diagram D.

The art of Kirby calculus is to decide whether two diagrams determine diffeomorphic 4-manifolds. According to the basic theorem of the subject, D and D' determines diffeomorphic closed 4-manifolds if and only if they can be transformed into each other by a finite sequence of isotopies, handle slides and handle pair creation/cancellation.

1.

2. The solution to this problem is given in detail in the solution of Problem 1 of Tuesday (by Inanc Baykur).

3. (Vera Vertesi and Raphael Zentner 1st & 2nd, Khaled Qazaqzeh 3rd)

   Our intention is to show that the two forms $q_1 := E_8 \oplus \langle -1 \rangle$ and $q_2 := 9\langle -1 \rangle$ on $\mathbb{Z}^9$ are not equivalent, that is, there is no isomorphism of $\mathbb{Z}^9$ that turns one form into the other. Here the convention is such that the form E_8 has signature equal to -8.

   1st solution: We argue by considering characteristic elements of these forms. An element z is called a characteristic element of the form q if one has

$$q(z, x) \equiv q(x, x) \pmod{2\mathbb{Z}}$$

   for all elements x of the underlying module.

   Let us consider the diagonal form q_2. We express a characteristic element z_2 in the canonical basis (e_i) of $\mathbb{Z}^9$,

$$z_2 = \sum_{i=1}^{9} a_i \, e_i \, .$$

   We see that the coefficients a_i all have to be odd numbers, as their pairing with the basis elements has to yield odd numbers. Therefore the highest possible square of a characteristic element z_2 is -9, obtained by the characteristic element $\sum e_i$.

On the other hand, the form E_8 is even, so the element 0 is a characteristic element of E_8, and so a characteristic element of q_1 is given by $z_1 = e_9$. This has square -1. Therefore, the two elements are not equivalent.

2nd solution: A shorter solution that Andras Stipsicz told us about is this: One counts the number of elements with square -1. The form $q_1 = E_8 \oplus \langle -1 \rangle$ has only two such elements, namely the elements $\pm e_9$, whereas the form q_2 has 18 elements with square -1.

3rd solution: Let λ_1, λ_2 be the forms $E_8 \oplus < -1 >$ and $9 < -1 >$ respectively. Let us verify all the required properties.

(a) Let $x = (0,0,0,0,0,0,0,0,1)$. Then $\lambda_1(x,x) = \lambda_2(x,x) = 1$. Therefore the two forms are both odd.

(b) It is clear that $\sigma(\lambda_1) = -9 = \sigma(\lambda_2)$.

(c) It is also clear that $\text{rank}(\lambda_1) = 9 = \text{rank}(\lambda_2)$.

These two forms are not equivalent since if they are we get two different decompositions of the same definite form as a result of the following proposition and this contradicts the following theorem due to Eichler (see [M-H, Theorem 6.4]).

Theorem 3.1. *Every positive definite inner product space over $\mathbb{Z}$ splits uniquely as an orthogonal sum of indecomposable spaces.*

Now up to isometry there is only one negative definite form of rank 8, [M-H, Remark 1. p. 28] we have the following

Proposition 3.2. *The bilinear form E_8 is indecomposable.*

4. (AS)

The handle decomposition naturally induces a CW decomposition of the underlying topological space (since handles are 'thickened' cells). Therefore the 1-handles correspond to the generators, while the 2-handles to the relations of the corresponding presentation of the fundamental group. More concretely, suppose that there are $m = |D_1|$ dotted circle. Let the generator x_i correspond to the i^{th} unknot in D_1. Fix spanning oriented disks for the unknots in D_1, and for the 2-handle attached along the knot $K \in D - D_1$ fix a starting point and a direction on the knot. The relation r_K corresponding to K is the word we get by traversing along K in the given direction and writing down x_i^ϵ everytime we intersect the spanning disk of the unknot corresponding to x_i. (The exponent ϵ is determined by the sign of the intersection.)

In the same way, in computing the homology of $X(D)$ we use CW-homology. Let C_i denote the free Abelian group generated by the i-handles of the handle decomposition. Obviously, $H_i(X(D);\mathbb{Z}) = 0$ for $i > 2$ and it is $\mathbb{Z}$ for $i = 0$. Let $\varphi\colon C_2 \to C_1$ be defined on $K \in D - D_1$ by sending it to

$$\sum_{i=1}^{m} \ell k(K, N_i)$$

where $D_1 = \{N_1, \dots N_m\}$ is the set of dotted circles in D. Then the kernel of φ is isomorphic to $H_2(X(D); \mathbb{Z})$ and $C_1/Im\,\varphi$ gives $H_1(X(D); \mathbb{Z})$.

Notice that in the above discussion the framings played no role. In the simple case when $D_1 = \emptyset$ and $Y(D) = S^3$ (that is, when the 4-manifold $Z(D)$ admits a handle decomposition without 1- and 3-handles), then $H_2(Z(D); \mathbb{Z}) \cong H^2(Z(D); \mathbb{Z})$ is isomorphic to the free Abelian group generated by the 2-handles, and the linking matrix of D (with the framings in the diagonal) provide the *intersection form*, that is, the ring structure on $H^*(Z(D); \mathbb{Z})$.

5. (Ahmet Beyaz)

   (a) This 4-manifold with boundary is like a thickened 2-dimensional 2-handle. i.e. a disk bundle over the 2-sphere. Let us calculate the Euler class. If we push the interior of the disk enclosed by the unknot to the interior of D^4 we can see it can be closed by the core of the 2-handle we attach to it to form a sphere S^2. Now take the copy of this sphere and push it along the unknot linked to the unknot in the Kirby diagram n times. These spheres will intersect each other at the points where the second unknot intersect the pushed disk intersects the framing unknot. This intersection number is the Euler number evaluated on the fundamental class of the base manifold which gives the Euler class of the disk bundle.

   (b) By the above discussion, the boundary of $X(D)$ for this particular D is $S^1 \times S^2$, hence we can attach to $X(D)$ a 3- and a 4-handle and get a closed 4-manifold. The 2-handle attached along the 0-framed unknot, together with the 3-handle provides a cancelling pair, hence we do not change the diffeomorphism type of $Z(D)$ by deleting these two handles. The resulting empty diagram D_0 then provides D^4 as $X(D_0)$ and the 4-sphere S^4 as $Z(D_0)$.

   (c) A single unknot with framing 1 and -1 respectively. The boundary of the resulting 4-manifold is S^3, hence it can be capped by a D^4.

   (d) This is S^4 because the attaching sphere of the 2-handle intersects the co-core of the 1-handle once, so they cancel and there remains an empty Kirby diagram.

6. (MK & DC) For (a) and (b) we apply obstruction theory and attempt to lift the map $M \to BO\langle 2k+1\rangle$ further over $EO\langle 2k+1\rangle$: such a lift is equivalent to a stable framing of TM. The obstructions to lifting lie in $H^{i+1}(M; \pi_i(O))$ for $i \geq 2k$. Since $M \to BO\langle 2k+1\rangle$ is $2k$-connected we see that M is $(2k-1)$-connected. In case (a) the boundary of M is a homotopy sphere Σ and so by Poincaré duality

$$H^{i+1}(M; \pi_i(O)) \cong H_{4k-i-1}(M, \Sigma; \pi_i(O)) = 0$$

   for $i \geq 2k$. It follows that M is stably parallelisable and since the boundary of M is non-empty it follows that M is parallelisable. This last point is a general fact for any compact manifold and is proven with similar sorts of obstruction theory arguments using the fibrations $S^n \to BO(n+1) \to BO(n)$. See for example [K-M, Lemma 3.5].

In case (b) there is a single non-zero group containing the obstruction to trivialising TM: it is $H^{4k}(M; \pi_{4k-1}(O(4k)) \cong \pi_{4k-1}(O(4k))$. In this case we can modify the tangent bundle of M by the action of a vector bundle ξ over S^{4k} to make this obstruction trivial. It follows that TM is isomorphic to $c^*\xi$.

We now move to part (c). Write $M = M^\bullet \cup_{S^{4k-1}} D^{4k}$ where $M^\bullet = M - \mathrm{Int}(D^{4k})$. By part (a) we know that there is a trivialisation

$$\alpha\colon TM|_{M^\bullet} \cong M^\bullet \times \mathbb{R}^{4k}.$$

If we compare the trivialisation α restricted to $S^{4k-1} = \partial M^\bullet = \partial D^{4k}$ with the standard trivialisation of TD^{4k} restricted to S^{4k-1} we obtain a map $\phi\colon S^{4k-1} \to O$. Looking at part (b) we see that we may identify the functions ϕ and ξ. But by the definition of the J-homomorphism, $J(S(\xi))$ is the framed bordism class of (S^{4k-1}, ξ) and we see that $(M^\bullet, \alpha)$ is a framed null-bordism of (S^{4k-1}, ξ). Hence $J(S(\xi)) = 0$.

7. (Tamás Terpai)

First we show that the statement of Thursday Problem 6b is valid for M. Indeed, let N be M with a small 4-ball cut out. This is still a spin manifold, and its virtual normal bundle is induced from $B\mathrm{Spin} = BO\langle 4\rangle$, a 3-connected space; but the homotopy type of N is that of a 3-dimensional cell complex, so the classifying map is nullhomotopic and the stable normal bundle of N is trivial. Hence the tangent bundle of N is stably trivial as well. But it is already stable when considered over a 3-skeleton of N (a 4-dimensional bundle over a 3-dimensional cell complex), and this latter can be chosen to be a deformation retract of N, hence the tangent bundle TN is trivial. As a consequence, TM is the pullback $c^*\xi$ of a vector bundle ξ on $M/N \cong S^4$, as claimed. Note that c has degree 1.

Now by Fact C (Hirzebruch's Signature Theorem) we have

$$\sigma(M) = \langle L_1(p_1(TM)), [M]\rangle = \langle L_1(c^* p_1(\xi)), [M]\rangle = \langle \frac{p_1(\xi)}{3}, c_*[M]\rangle = \frac{1}{3}\langle p_1(\xi), [S^4]\rangle.$$

By Problem 6c and Fact A we have that ξ is identified with some multiple of 24 in $\pi_3(SO) \cong \mathbb{Z}$. Fact B (Bott, Kervaire) tells us that $p_1(\xi)$ has to be a multiple of $24a_1 = 48$, and hence $\sigma(M) = \frac{1}{3}\langle p_1(\xi), [S^4]\rangle$ is divisible by $48/3 = 16$.

8. No solution.

9. (Anna Abczynski)

Let M_0 and M_1 be closed smooth 4-connected 10-manifolds with equal Euler characteristic. We will use the fact that the natural homormophism $\Psi : \Theta_{10} \to \Omega_{10}^{\mathrm{String}} = \mathbb{Z}/6$ is an isomorphism.

(a) Observe that the 4-type of M_0 and M_1 is $B\mathrm{String} = BO\langle 8\rangle$ since they are both 4-connected whence $\frac{p_1}{2}$ and w_2 vanish that is, the normal Gauss map admits a lift to $B\mathrm{String}$. Since by assumption $\chi(M_0) = \chi(M_1)$ we know that M_0 and M_1 are

stably diffeomorphic, i.e. $M_0\sharp_r(S^5 \times S^5) \cong M_1\sharp_r(S^5 \times S^5)$, if and only if they are BString-bordant. By [K, Corollary 7.4] they are even diffeomorphic.
From the hint we know that $\Omega_{10}^{\text{String}} = \mathbb{Z}/6$ is isomorphic to the group of exotic 10-spheres. The group structure is given by connected sum. Hence there always exists $\Sigma \in \Theta_{10} \cong \Omega_{10}^{\text{String}}$ such that M_0 and $M_1\sharp\Sigma$ are string-bordant that is diffeomorphic. Note that we use here that the connected sum with a homotopy sphere does not change the Euler characteristic.

(b) Since the image of standard sphere under under Ψ represents the trivial element in $\Omega_{10}^{\text{String}}$ we get $[M] \neq [M\sharp\Sigma] \in \Omega_{10}^{\text{String}}$ if $\Psi^{-1}(\Sigma) \in \Theta_{10} - \{[S^{10}]\}$. Therefore they are not diffeomorphic as string manifolds. By obstruction theory, each 4-connected 10-manifold has a unique string structure and so we conclude that M and $M\sharp\Sigma$ are not diffeomorphic.

(c) From the first part of the exercise we know that there always exists an element in $\Sigma \in \Theta_{10}$ such that $[M] \cong [M\sharp\Sigma]$; from the second part we know it cannot be an element in $\Theta_{10} - \{[S^{10}]\}$.

(d) We have $S^{10}\sharp_l\Sigma$ and $(\sharp_r S^5 \times S^5)\,\sharp_l\Sigma$, $l \leq 5$ where Σ now denotes a generator of Θ_{10}.

4 Friday

1. (Daniel Kasprowski)

   Define $F : \mathcal{S}(M) \to \mathcal{M}(M)$ by $[f : N \to M] \mapsto [N]$. Obviously F is onto and the action of $\mathcal{E}(M)$ preserves the fibers of F.
   If $F([f_0 : N \to M]) = F([f_1 : N \to M])$ then $f_0^{-1} \circ f_1$ is an element of $\mathcal{E}(M)$ such that
   $$[f_0 \colon N \to M] \cdot (f_0^{-1} \circ f_1) = [f_1 \colon N \to M].$$
   It follows that $[[f_0 : N \to M]] = [[f_1 : N \to M]] \in \mathcal{S}(M)/\mathcal{E}(M)$ and so the map $F : \mathcal{S}(M)/\mathcal{E}(M) \to \mathcal{M}(M)$ is a bijection.

2. No solution.

3. No solution.

4. No solution.

5. No solution.

6. No solution.

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diarmuidc23@gmail.com
http://www.dcrowley.net/teaching.html